Very free R-equivalence on toric models

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Abstract

Using the theory of the universal torsor, we prove that two rational points on a smooth projective toric variety over an infinite field that are rationally equivalent can in fact be connected by a very free rational curve. We also show a similar result over del Pezzo surfaces of degree 5.

Introduction

Let X be a smooth projective variety over an infinite field k, and assume that X is (geometrically) *separably rationally connected*, meaning that, over the algebraic closure $\bar{k}$, there exists an $f\colon \mathbb{P}^1_{\bar{k}} \to X_{\bar{k}}$ which is "very free" in the sense that f^*T_X is ample (in other words, $H^1(\mathbb{P}^1, (f^*T_X)(-2)) = 0$). If x and y are two k-rational points of X (assuming there are any) which are "R-equivalent", that is, which can be joined by a chain of rational curves on X (each defined over k), we can ask ourselves whether there exists $f\colon \mathbb{P}^1_k \to X$ defined over k such that $f(0) = x$, and $f(\infty) = y$ and $H^1(\mathbb{P}^1, (f^*T_X)(-2)) = 0$: if such is the case, we say that x and y are R-equivalent by a single very free rational curve.

It is true over k algebraically closed that any two points on a smooth projective separably rationally connected variety X are in fact joined by a single very free rational curve (see [7] for a proof of this fact as well as all introductory material on rationally connected varieties).

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In the case where k is no longer algebraically closed but "large", meaning that every irreducible variety that has a smooth k-point has a Zariski dense subset of them, for example when k is a local field, then the answer is again affirmative: this is the result of a recent work by Kollár ([8], theorem 23)—any two points (on a smooth projective rationally connected variety) which are R-equivalent are so by a single very free rational curve.

For other fields k, however, the answer to the question is unknown, even in some simple cases.

When X is a smooth del Pezzo surface of degree 4 over k, for example, it is known that every universal torsor over X (a term which we will define below) having a k-point is k-rational (see [3]), so two rationally equivalent points on X are R-equivalent, but it is not known whether they can be joined by a single very free rational curve. Perhaps more to the point, it can be shown, using the technique of the present paper, that, for every R-equivalence class α of $X(k)$, there is a nonempty Zariski open set U_α of X such that if P and Q are in α and in U_α then they are joined by a single very free rational curve—but it remains unknown whether, in fact, U_α can be taken to be X.

A positive answer to the question in full generality (for any infinite field k and any separably rationally connected variety X) is conceivable, but seems out of reach with present techniques.

In this article we prove a positive result when X is a toric model (i.e., a smooth equivariant compactification of a torus) over an infinite field k: this is possible because a universal torsor can be explicitly constructed, and because rational curves can be moved thanks to the action of the torus. In the next section, we also prove a positive result in the case where X is a del Pezzo surface of degree 5 (another case in which the universal torsor is well controlled). We start with some general remarks on very free R-equivalence.

1 General framework

We introduce a notation: if k is a field and X an irreducible variety over k, and if $x, y \in X(k)$, let $x \overset{X}{\leftrightarrow} y$ stand for the following statement: there exists an irreducible variety M over k such that $M(k)$ is Zariski-dense in M, and a dominant and separable rational map $F\colon M \times \mathbb{P}^1 \dashrightarrow X$ such that F restricted to $M \times \{0\}$ is constant equal to x and F restricted to $M \times \{\infty\}$ is constant equal to y.

The following proposition summarizes some general facts about this relation:

Proposition 1. *Assume k is a field and X is an irreducible variety over k. Then:*

1. *If $U \subseteq X$ is a Zariski open set and $x, y \in U(k)$ then $x \overset{X}{\leftrightarrow} y$ if and only if $x \overset{U}{\leftrightarrow} y$.*

2. *If $X = \mathbb{P}^n_k$ then $x \overset{X}{\leftrightarrow} y$ for any two x, y.*

3. *Suppose $p \colon Z \dashrightarrow X$ is a dominant and separable rational map with Z an irreducible variety over k: then, for any $x, y \in Z(k)$ at which p is defined, if $x \overset{Z}{\leftrightarrow} y$ then $p(x) \overset{X}{\leftrightarrow} p(y)$.*

4. *Suppose X is smooth projective: then, for any $x, y \in X(k)$, if $x \overset{X}{\leftrightarrow} y$ then x and y are R-equivalent by a single very free rational curve.*

The first fact is trivial (restrict F to U on the range).

To prove the second, consider $x, y \in \mathbb{P}^n_k(k)$ and take the family of all smooth conics passing through x and y and parameterize them rationally: obviously we can find an open set M in some affine space over k (so certainly $M(k)$ is dense) and a dominant and separable morphism $F \colon M \times \mathbb{P}^1 \to \mathbb{P}^n$ which takes $M \times \{0\}$ to x and $M \times \{\infty\}$ to y.

The third statement is trivial: merely compose F with p.

To get the fourth, first notice that when X is projective we can by restricting M assume that F is a morphism; now apply the following geometric result (see, e.g., [7], II.3.10):

Proposition 2. *Let $\bar{k}$ be an algebraically closed field, M an irreducible variety over $\bar{k}$, and X a smooth projective variety over $\bar{k}$. Let $x \in X$. Finally, let $F \colon M \times \mathbb{P}^1 \to X$ be a separable and dominant morphism such that $F(M \times \{0\}) = \{x\}$. Then there exists a* nonempty *Zariski open set M^0 of M such that for all $p \in M^0$ the morphism $F_p \colon \mathbb{P}^1 \to X$ satisfies the condition that $F_p^* T_X$ be ample.*

—and make use of the fact that M^0 has a point over k by assumption.

2 R-equivalence on universal torsors

The goal of this section is to prove the following result:

Proposition 3. *Let T be an algebraic torus over an infinite field k, and X a smooth equivariant compactification of T; then given two k-rational points x, y of X, if x and y are rationally equivalent, they are R-equivalent by a single very free rational curve.*

To do this, we use the following result, whose proof will be given in the appendix:

Proposition 4. *Let T be an algebraic torus on a field k, and X a smooth equivariant compactification of T; then there exists a torus S over k, a "universal" S-torsor $p\colon \mathscr{T} \to X$, and an S-equivariant open embedding of $\mathscr{T}$ in an affine space on which S acts linearly.*

"Universal" is to be taken in the sense of [1], II.C (or [2], example 2.3.3), which we presently recall. Call $H^1(X, S)$ the étale cohomology group classifying S-torsors on X, and $[\mathscr{T}]$ the class of p in it. Define a map $\chi\colon H^1(X,S) \to \operatorname{Hom}_{\operatorname{Gal}(\bar{k}/k)}(S^*, \operatorname{Pic}\bar{X})$ which sends the class of an S-torsor on X, say $\mathscr{S}$, and a character $\lambda \in S^* = \operatorname{Hom}(\bar{S}, \bar{\mathbb{G}}_m)$ to the class of the $\bar{\mathbb{G}}_m$-torsor on $\bar{X}$ deduced from $\bar{\mathscr{S}}$ by λ. To say that $\mathscr{T}$ is universal means that $S^* = \operatorname{Pic}\bar{X}$ and that $\chi([\mathscr{T}])$ is the identity on $\operatorname{Pic}\bar{X}$.

We will need the following fact:

Lemma 5. *Let T and X be as in proposition 3, and let $p\colon \mathscr{T} \to X$ be a universal torsor on X. Then there exists a point $z \in T(k)$ such that the class $[\mathscr{T} \times_X \operatorname{Spec} k_z] \in H^1(k, S)$ of the fiber of $\mathscr{T}$ over z is trivial, i.e. $\mathscr{T}$ has a k-point over z.*

Proof. Let $\alpha = [\mathscr{T} \times_X \operatorname{Spec} k_o] \in H^1(k,S)$ be the class of the fiber of $\mathscr{T}$ over the origin $o \in T(k)$. Let $\mathscr{T}^o$ be the torsor defined by $[\mathscr{T}^o] = [\mathscr{T}] - \alpha$: then $\mathscr{T}^o$ is the universal torsor that is trivial[1] above o, and, from the discussion in [1], III (see also [2], 2.4.4), the map $T(k) \to H^1(k,S)$, $z \mapsto [\mathscr{T}^o \times_X \operatorname{Spec} k_z]$ is surjective. In particular, there exists z such that $[\mathscr{T}^o \times_X \operatorname{Spec} k_z] = -\alpha$, so $[\mathscr{T} \times_X \operatorname{Spec} k_z] - \alpha = -\alpha$, which proves that $[\mathscr{T} \times_X \operatorname{Spec} k_z]$ is nil, what we wanted. □

[1] In fact, if the torsor $\mathscr{T}$ is that which we shall construct in the appendix, it is easy to see that it is already the universal torsor trivial over o; however, we shall not use this fact, which only very slightly simplifies the proof.

Now apply lemma 5 to the universal torsor $\mathscr{T}$ given by proposition 4: we see that there exists $z' \in T(k)$ such that the fiber of $\mathscr{T}$ over z' is trivial. Apply now the same lemma to the universal torsor $\mathscr{T}^x$ with trivial fiber over x (in other words the torsor given by $[\mathscr{T}^x] = [\mathscr{T}] - [\mathscr{T} \times_X \operatorname{Spec} k_x]$): so there exists $z \in T(k)$ such that the fiber of $\mathscr{T}^x$ over z is trivial. Let $\tau_{z'-z} \colon X \to X$ be the translation by $z' - z$: the torsor $\tau^*_{z'-z}\mathscr{T}$ is still universal (since $\tau_{z'-z}$ acts trivially on $\operatorname{Pic} \bar{X}$) and it is trivial over z—therefore it is isomorphic to $\mathscr{T}^x$ (which has the same property).

Let $x' = \tau_{z'-z}(x)$ and $y' = \tau_{z'-z}(y)$. Since $\mathscr{T}^x \cong \tau^*_{z'-z}\mathscr{T}$ is trivial over x, it follows that $\mathscr{T}$ is trivial over x'. But, since y is rationally equivalent to x by [1], II.B, proposition 1, $\mathscr{T}^x$ is also trivial over y, and therefore so is $\mathscr{T}$ over y'. So there exist points P and Q of $\mathscr{T}(k)$ over x' and y' respectively, and proposition 4 shows that P and Q live inside an open set of an affine space $\mathbb{A}$ over k.

Finally, using the general facts laid out in proposition 1 (1–4), we have $P \overset{\mathbb{A}}{\leftrightarrow} Q$ (use facts 1–2) so $P \overset{\mathscr{T}}{\leftrightarrow} Q$ (fact 1 again) and therefore $x' \overset{X}{\leftrightarrow} y'$ (fact 3: compose with p) so $x \overset{X}{\leftrightarrow} y$ (compose with $\tau_{z-z'}$) which gives the desired conclusion (from fact 4). □

3 Del Pezzo surfaces of degree 5

We now turn to the case where X is a del Pezzo surface of degree 5 over k. Then it is known that there is a unique universal torsor $p \colon \mathscr{T} \to X$ on X ("unique" up to non-unique isomorphism), trivial over every point, and that it is an open set of the Grassmanian variety $\operatorname{Gr}(2,5)$ of lines in $\mathbb{P}^4$ (Skorobogatov, [11], theorem 3.1.4).

If now x and y are two arbitrary k-rational points on X, pick k-rational points in $p^{-1}(x)$ and $p^{-1}(y)$ (which exist because $\mathscr{T}$ is trivial over x and y), corresponding to two lines Δ and Λ in $\mathbb{P}^4$. Now let Π and Π' be two hyperplanes in $\mathbb{P}^4$ neither of which contains either Δ or Λ and such that the intersection points P, P' of Π, Π' with Δ are distinct and similarly for the intersection points Q, Q' of Π, Π' with Λ. Then we have a rational map $\Pi \times \Pi' \dashrightarrow \mathscr{T} \to X$ taking a point on Π and one on Π' to the line they define (in general) and then to the image point by p in X. Again by the general facts laid out in proposition 1, since $(P, P') \overset{\Pi\times\Pi'}{\leftrightarrow} (Q, Q')$, we get $x \overset{X}{\leftrightarrow} y$ and consequently x and y are R-equivalent by a single very free rational curve.

Thus, we have shown:

Proposition 6. *Let X be a del Pezzo surface of degree 5 over an infinite field k; then given any two k-rational points x, y of X, there exists $f\colon \mathbb{P}^1_k \to X$ such that $f(0) = x$ and $f(\infty) = y$ with, further, f^*T_X ample.*

Appendix: Explicit construction of a universal torsor over a toric variety

Proposition 4 remains to be settled. A proof can be found in [10] (proposition 8.5), but the one we give below, for the reader's convenience, seems much more straightforward.

Historical remark: The construction described here was introduced in [5] and [4]. Here we give a presentation similar to the one contained in [9], although universality of the torsor is not shown there.

Let $T^* = \mathrm{Hom}_{\bar{k}}(\bar{T}, \bar{\mathbb{G}}_m)$ be the lattice of characters of the torus T, and $T_* = \mathrm{Hom}_{\bar{k}}(\bar{\mathbb{G}}_m, \bar{T})$ the lattice, dual to the former, of cocharacters. One and the other are endowed with an action of the Galois group $\Gamma = \mathrm{Gal}(\bar{k}/k)$. We write $T^*_{\mathbb{R}} = T^* \otimes_{\mathbb{Z}} \mathbb{R}$ for the real vector space in which T^* lives, and $T_{*\mathbb{R}} = T_* \otimes_{\mathbb{Z}} \mathbb{R}$ for the real vector space, dual to the former, in which T_* lives. The general theory of toric varieties (cf. [6], in particular §2.3) allows us to describe X by means of a fan Σ of strongly convex rational polyhedral cones in $T_{*\mathbb{R}}$. The fact that X is smooth means (cf. [6], §2.1) that every cone $\sigma \in \Sigma$ is spanned by part of a basis of T_*, determined uniquely by σ: call B_σ the part in question, and let $P = \bigcup_{\sigma\in\Sigma} B_\sigma$ be the union of the B_σ for all $\sigma \in \Sigma$. Then P is a finite part of T_* which spans the latter and is stable under the action of Γ. For every $\sigma \in \Sigma$, we have $B_\sigma = \sigma \cap P$, and σ is spanned by $\sigma \cap P$.

Now let V_* be the (free) lattice with basis P (with the obvious action of Γ making it a permutation lattice), and V^* the dual lattice, and $V_{*\mathbb{R}}$ and $V^*_{\mathbb{R}}$ the real vector spaces in which they respectively live. We call V the dual torus to V^* (i.e. the torus of which V^* is the character lattice), so $\bar{V} = \mathrm{Spec}\, \bar{k}[z^u : u \in V^*]$: since V^* is a permutation lattice, V is a quasi-trivial torus. And let $\mathbf{A}$ be the affine space defined by the cone of $V_{*\mathbb{R}}$ spanned by the elements of P. Since P spans T_*, we have a surjective morphism $V_* \to T_*$ and thus an injection $T^* \to V^*$.

From the description in [6], §3.3, the lattice V^* is precisely the group $\mathrm{Div}_{\bar{X}\setminus\bar{T}}\,\bar{X}$ of $\bar{T}$-invariant divisors of $\bar{X}$, by the arrow which sends a $u \in V^*$ to $\sum_{p\in P} u(p)D_p$ (where D_p is the closure of the orbit of $\bar{T}$ acting on $\bar{X}$ associated to the ray spanned by p in $V_{*\mathbb{R}}$). With this identification, $T^* \to V^*$ sends a $u \in T^*$ to the principal divisor $\mathrm{div}(t^u)$, and its cokernel ([6], §3.4) is the Picard group of $\bar{X}$, which is itself a lattice, say S^*, dual to a torus S. We therefore have the short exact sequence of lattices $0 \to T^* \to V^* \to S^* \to 0$, equal to $0 \to \bar{k}[\bar{T}]^\times/\bar{k}^\times \to \mathrm{Div}_{\bar{X}\setminus\bar{T}}\,\bar{X} \to \mathrm{Pic}\,\bar{X} \to 0$, and the dual short exact sequence of tori $1 \to S \to V \to T \to 1$.

For every cone $\sigma \in \Sigma$, let $\sigma^\vee = \{u \in T^*_{\mathbb{R}} : (\forall v \in \sigma)(\langle u, v\rangle \geq 0)\}$ denote the dual cone, and let $\bar{X}(\sigma) = \mathrm{Spec}\,\bar{k}[t^u : u \in T^* \cap \sigma^\vee]$ be the spectrum of the semigroup algebra of $T^* \cap \sigma^\vee$: thus, $\bar{X}$ is obtained precisely by gluing the $\bar{X}(\sigma)$ for $\sigma \in \Sigma$ (identifying the open set $\bar{X}(\sigma \cap \sigma')$ in $\bar{X}(\sigma)$ and $\bar{X}(\sigma')$). Similarly, given a cone $\sigma \in \Sigma$, which is, therefore, spanned by a finite set (called B_σ) of elements of P, we can consider the cone $\tilde{\sigma}$ in $V_{*\mathbb{R}}$ spanned by the same elements of P, and its dual $\tilde{\sigma}^\vee$, a cone in $V^*_{\mathbb{R}}$: let us call $\bar{\mathbf{A}}(\sigma) = \mathrm{Spec}\,\bar{k}[z^u : u \in V^* \cap \tilde{\sigma}^\vee]$ the spectrum of the corresponding semigroup algebra. Thus $\bar{\mathbf{A}}(\sigma)$ is an open set in $\bar{\mathbf{A}}$, containing $\bar{V}$. Furthermore, the inclusion $T^* \to V^*$, which manifestly sends $T^* \cap \sigma^\vee$ inside $V^* \cap \tilde{\sigma}^\vee$, defines a morphism $\bar{\mathbf{A}}(\sigma) \to \bar{X}(\sigma)$.

To make the situation clearer, let us presently prove the following lemma (lemma 5.1 of [9]):

Lemma 7. *Let $\delta \in S^*$ and let $\sigma \in \Sigma$. Then there exists a $u_\delta \in V^*$ (not necessarily unique) which maps to $\delta \in S^*$ (by the arrow $V^* \to S^*$ defined above) and such that $\langle u_\delta, p\rangle = 0$ for all $p \in B_\sigma$ (in other words $u_\delta \in V^* \cap \tilde{\sigma}^\vee \cap (-\tilde{\sigma}^\vee)$).*

Proof. The morphism $V^* \to S^*$ being surjective, there exists $v \in V^*$ which maps to $\delta \in S^*$. Since B_σ is a subset of a basis of T_*, there exists $\tilde{v} \in T^*$ such that $\langle \tilde{v}, p\rangle = \langle v, p\rangle$ for all $p \in B_\sigma$. We then take $u_\delta = v - \tilde{v}$. □

A u_δ as given by the previous lemma defines a $z^{u_\delta} \in \bar{k}[z^u : u \in V^* \cap \tilde{\sigma}^\vee]$ which is invertible in this algebra, since $-u_\delta$ manifestly also belongs to $\tilde{\sigma}^\vee$. We deduce the following description: □

Fact 8. *$\bar{k}[z^u : u \in V^* \cap \tilde{\sigma}^\vee]$, seen as a module over $\bar{k}[t^u : u \in T^* \cap \sigma^\vee]$, is free and a basis is formed by invertible elements z^{u_δ}, one for each δ in S^*; the free sub-module of rank 1 corresponding to a δ in S^* is precisely the set of linear combinations of the z^u for those $u \in V^* \cap \tilde{\sigma}^\vee$ for which $u|_{S_*}$ (that is,* *the image of u by $V^* \to S^*$) is δ. This can also be expressed by saying that $\bar{k}[z^u : u \in V^* \cap \tilde{\sigma}^\vee]$ is graded by S^* as an algebra over $\bar{k}[t^u : u \in T^* \cap \sigma^\vee]$, each graded component containing an invertible element.*

In particular, we see that if $\sigma' \subseteq \sigma$ in Σ, the tensor product of $k[z^u : u \in V^* \cap \tilde{\sigma}^\vee]$ with $\bar{k}[t^u : u \in T^* \cap \sigma'^\vee]$ over $\bar{k}[t^u : u \in T^* \cap \sigma^\vee]$ is $k[z^u : u \in V^* \cap \tilde{\sigma}'^\vee]$, which means that the inverse image by $\bar{\mathbf{A}}(\sigma) \to \bar{X}(\sigma)$ of $\bar{X}(\sigma')$ is $\bar{\mathbf{A}}(\sigma')$, and, more precisely, that the morphism $\bar{\mathbf{A}}(\sigma') \to \bar{X}(\sigma')$ is exactly the restriction of $\bar{\mathbf{A}}(\sigma) \to \bar{X}(\sigma)$ to $\bar{X}(\sigma')$. The union of the $\bar{\mathbf{A}}(\sigma)$ for $\sigma \in \Sigma$, which we call $\bar{\mathscr{T}}$, comes from a variety $\mathscr{T}$ defined over k and open in $\mathbf{A}$, and by gluing we have a morphism $\mathscr{T} \to X$.

We also see that $\bar{k}[z^u : u \in V^* \cap \tilde{\sigma}^\vee]$ is faithfully flat over $\bar{k}[t^u : u \in T^* \cap \sigma^\vee]$. Thus, the morphism $\mathscr{T} \to X$ is faithfully flat. We get an action of V on $\mathscr{T}$ because $\mathscr{T}$ has been constructed as a toric variety (with cones $\tilde{\sigma} \subseteq V_{*\mathbb{R}}$); therefore, by restriction, we get an action of S on $\mathscr{T}$, which by construction leaves X invariant. To see that this gives us a torsor under S, it is enough to see that each $\bar{\mathbf{A}}(\sigma) \to \bar{X}(\sigma)$ is a torsor under $\bar{S}$. In other words, we must show that the morphism

$$\theta\colon \bar{S} \times \bar{\mathbf{A}}(\sigma) \to \bar{\mathbf{A}}(\sigma) \times_{\bar{X}(\sigma)} \bar{\mathbf{A}}(\sigma)\ , \quad (s,a) \mapsto (s \cdot a, a)$$

is an isomorphism. But the (co)morphism of the associated algebras from which it comes is given by

$$\begin{aligned}\theta^*\colon \bar{k}[z^u : u \in V^* \cap \tilde{\sigma}^\vee] \otimes_{\bar{k}[t^u : u \in T^* \cap \sigma^\vee]} \bar{k}[z^u : u \in V^* \cap \tilde{\sigma}^\vee] \\ \to \bar{k}[\chi^\lambda : \lambda \in S^*] \otimes_{\bar{k}} \bar{k}[z^u : u \in V^* \cap \tilde{\sigma}^\vee] \\ z^u \otimes z^{u'} \mapsto \chi^{u|_{S_*}} \otimes z^{u+u'}\end{aligned}$$

To see that this is indeed an isomorphism, notice that according to fact 8, the left-hand side has a basis over $\bar{k}[t^u : u \in T^* \cap \sigma^\vee]$ formed by the $z^{u_\delta} \otimes z^{u_{\delta'}}$ with u_δ as given in lemma 7, and the right-hand side has a basis formed by the $\chi^\lambda \otimes z^{u_{\delta''}}$. And on these two bases, the homomorphism in question is represented by a diagonal matrix whose coefficients are $t^{u_\delta + u_{\delta'} - u_{\delta''}}$ (for $\delta'' = \delta + \delta'$ and $\lambda = \delta$), which are invertible in $\bar{k}[t^u : u \in T^* \cap \sigma^\vee]$.

It remains to see that this torsor $p\colon \mathscr{T} \to X$ is indeed universal.

If $\sigma \in \Sigma$, since $\bar{X}(\sigma)$ is smooth, it is abstractly isomorphic to $\bar{\mathbb{A}}^d \times \bar{\mathbb{G}}_m^{n-d}$ (where d, say, is the dimension of σ and n that of T). In particular we have $\operatorname{Pic} \bar{X}(\sigma) = 0$; and furthermore $\bar{k}[t^u : u \in T^* \cap \sigma^\vee]^\times = \{t^u\colon u \in$

$T^*\cap\sigma^\vee\cap(-\sigma^\vee)\}$. The general exact sequence $0\to\bar{k}[\bar{U}]^\times/\bar{k}^\times\to\mathrm{Div}_{\bar{X}\setminus\bar{U}}\,\bar{X}\to\mathrm{Pic}\,\bar{X}\to 0$ (cf. [2], (2.3.10)) when $\mathrm{Pic}\,\bar{U}=0$ becomes, for $\bar{U}=\bar{X}(\sigma)$,

$$0\to T^*\cap\sigma^\vee\cap(-\sigma^\vee)\to V^*\cap\tilde{\sigma}^\vee\cap(-\tilde{\sigma}^\vee)\to S^*\to 0$$

The dual short exact sequence of tori is $1\to\bar{S}\to\bar{M}_\sigma\to\bar{R}_\sigma\to 1$, where $\bar{R}_\sigma$ and $\bar{M}_\sigma$ are quotients of $\bar{T}$ and $\bar{V}$ respectively. Furthermore, the quotient morphism $\bar{T}\to\bar{R}_\sigma$ extends to $\bar{X}(\sigma)$ (of which $\bar{T}$ is an open set): precisely, the morphisms $\bar{T}\to\bar{X}(\sigma)\to\bar{R}_\sigma$ give, on the associated algebras,

$$\bar{k}[t^u : u\in T^*\cap\sigma^\vee\cap(-\sigma^\vee)]\to\bar{k}[t^u : u\in T^*\cap\sigma^\vee]\to\bar{k}[t^u : u\in T^*]$$

By corollary 2.3.4 of [2], it is now sufficient to prove that $\bar{\mathbf{A}}(\sigma)\to\bar{X}(\sigma)$ is obtained as the pullback of $\bar{M}_\sigma\to\bar{R}_\sigma$ by the arrow $\bar{X}(\sigma)\to\bar{R}_\sigma$, moreover in a way compatible with the restrictions when $\sigma'\subseteq\sigma$. In other words, we are to determine (in a natural way) the fiber product $\bar{M}_\sigma\times_{\bar{R}_\sigma}\bar{X}(\sigma)$; this is the affine scheme whose algebra is the tensor product

$$\bar{k}[z^u : u\in V^*\cap\tilde{\sigma}^\vee\cap(-\tilde{\sigma}^\vee)]\otimes_{\bar{k}[t^u:u\in T^*\cap\sigma^\vee\cap(-\sigma^\vee)]}\bar{k}[t^u : u\in T^*\cap\sigma^\vee]$$

But (from fact 8) $\bar{k}[z^u : u\in V^*\cap\tilde{\sigma}^\vee]$ is free over $\bar{k}[t^u : u\in T^*\cap\sigma^\vee]$ with basis $\{z^{u_\delta}\}$ for $\delta\in S^*$; and for precisely the same reasons, $\bar{k}[z^u : u\in V^*\cap\tilde{\sigma}^\vee\cap(-\tilde{\sigma}^\vee)]$ is free over $\bar{k}[t^u : u\in T^*\cap\sigma^\vee\cap(-\sigma^\vee)]$ with the same basis. That is to say that the above tensor product is (by the natural map) $\bar{k}[z^u : u\in V^*\cap\tilde{\sigma}^\vee]$, in other words that $\bar{M}_\sigma\times_{\bar{R}_\sigma}\bar{X}(\sigma)=\bar{\mathbf{A}}(\sigma)$ (naturally).

This shows that the torsor $p\colon\mathscr{T}\to X$, obtained by gluing these different $\bar{\mathbf{A}}(\sigma)\to\bar{X}(\sigma)$, is indeed universal.

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