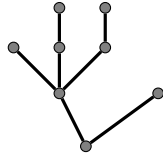
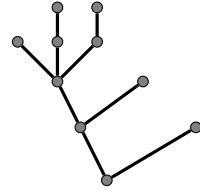


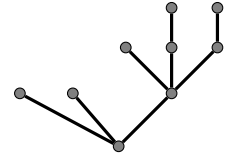
$$\phi_2(1) + 1 = \psi(\Omega 2) + 1$$



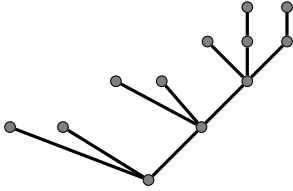
$$\phi_2(1) 2 = \psi(\Omega 2) 2$$



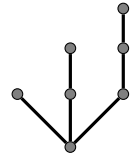
$$\phi_2(1)^{\phi_2(1)} = \psi(\Omega 2)^{\psi(\Omega 2)}$$



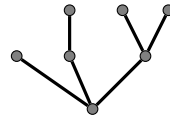
$$\phi_1(\phi_2(1) + 1) = \psi(\Omega 2 + 1)$$



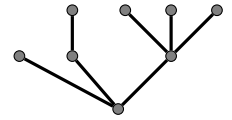
$$\phi_1(\phi_1(\phi_2(1) + 1)) = \psi(\Omega 2 + \psi(\Omega 2 + 1))$$



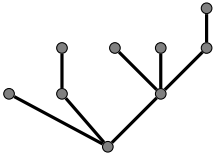
$$\phi_2(2) = \psi(\Omega 3)$$



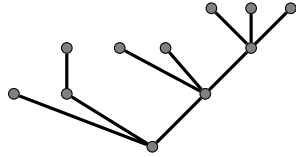
$$\phi_2(\omega) = \psi(\Omega \omega)$$



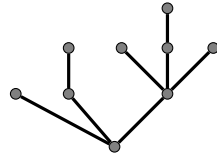
$$\phi_2(\varepsilon_0) = \phi_2(\phi_1(0)) = \psi(\Omega \psi(0))$$



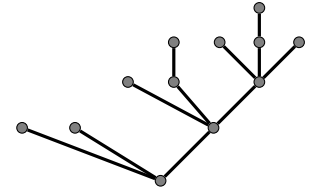
$$\phi_2(\varepsilon_1) = \phi_2(\phi_1(1)) = \psi(\Omega \psi(1))$$



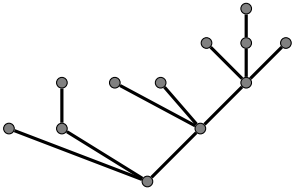
$$\phi_2(\phi_1(\phi_1(0))) = \psi(\Omega \psi(\psi(0)))$$



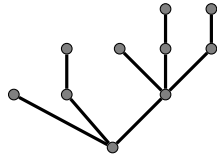
$$\phi_2(\phi_2(0)) = \psi(\Omega \psi(\Omega))$$



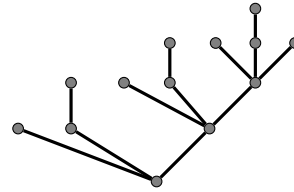
$$\phi_1(\phi_2(\phi_2(0)) + 1) = \psi(\Omega \psi(\Omega) + 1)$$



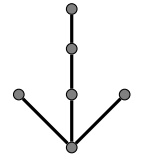
$$\phi_2(\phi_1(\phi_2(0) + 1)) = \psi(\Omega \psi(\Omega + 1))$$



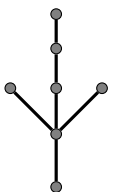
$$\phi_2(\phi_2(1)) = \psi(\Omega \psi(\Omega 2))$$



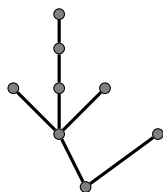
$$\phi_2(\phi_2(\phi_2(0))) = \psi(\Omega \psi(\Omega \psi(\Omega)))$$



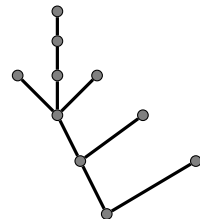
$$\phi_3(0) = \psi(\Omega^2)$$



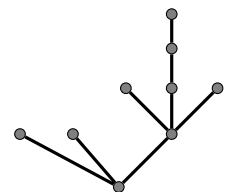
$$\phi_3(0) + 1 = \psi(\Omega^2) + 1$$



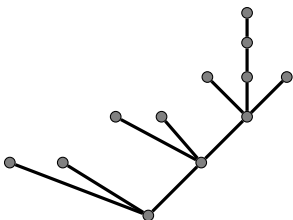
$$\phi_3(0) 2 = \psi(\Omega^2) 2$$



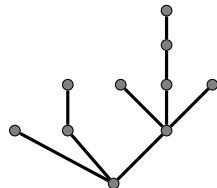
$$\phi_3(0)^{\phi_3(0)} = \psi(\Omega^2)^{\psi(\Omega^2)}$$



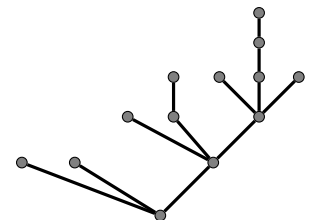
$$\phi_1(\phi_3(0) + 1) = \psi(\Omega^2 + 1)$$



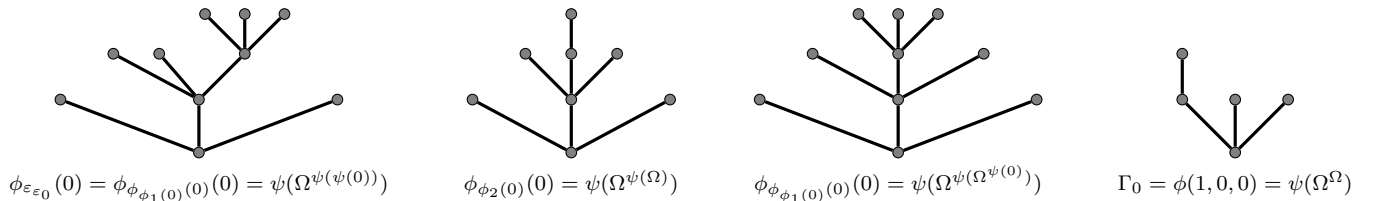
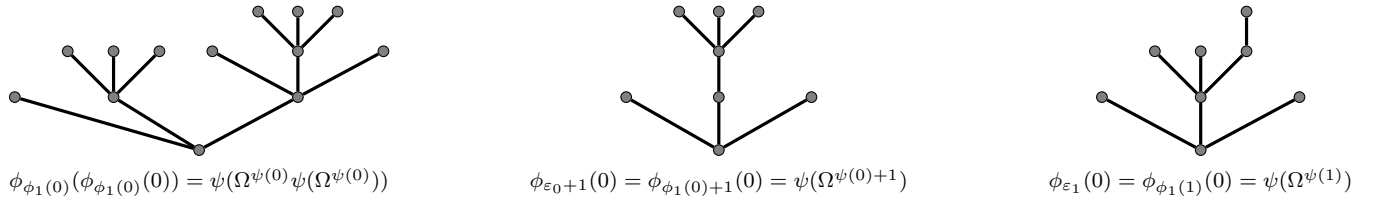
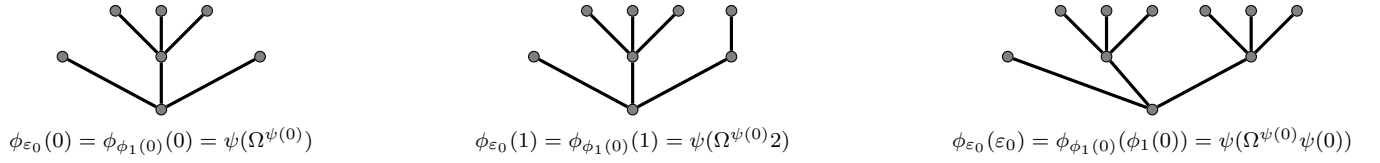
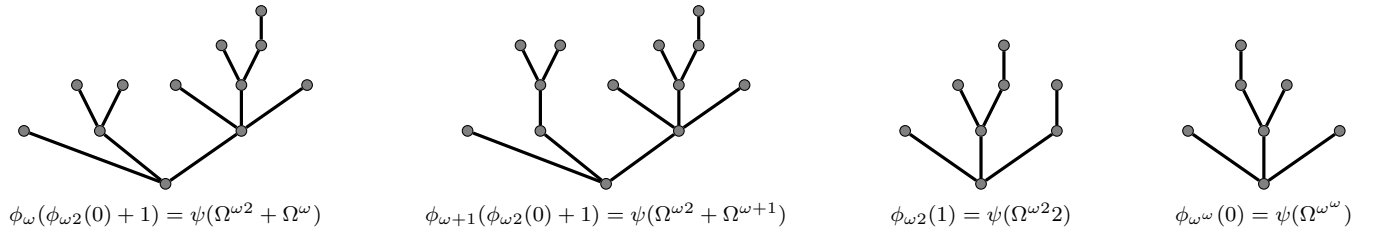
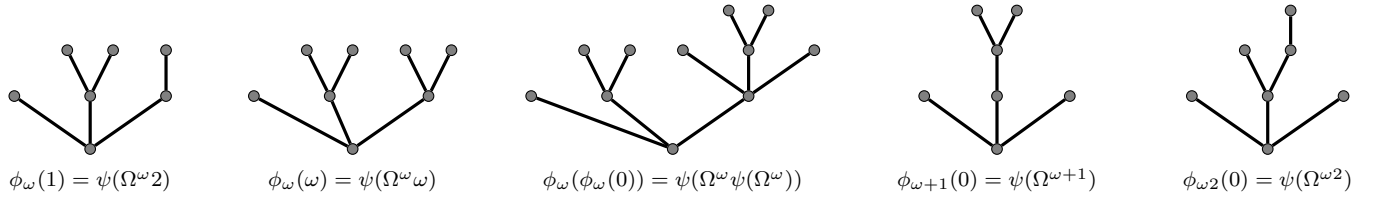
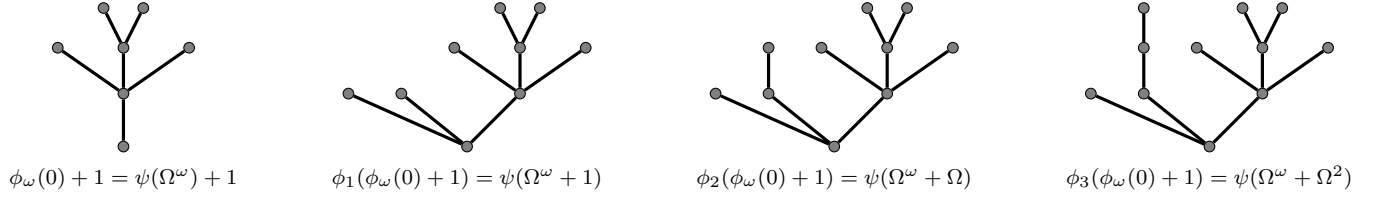
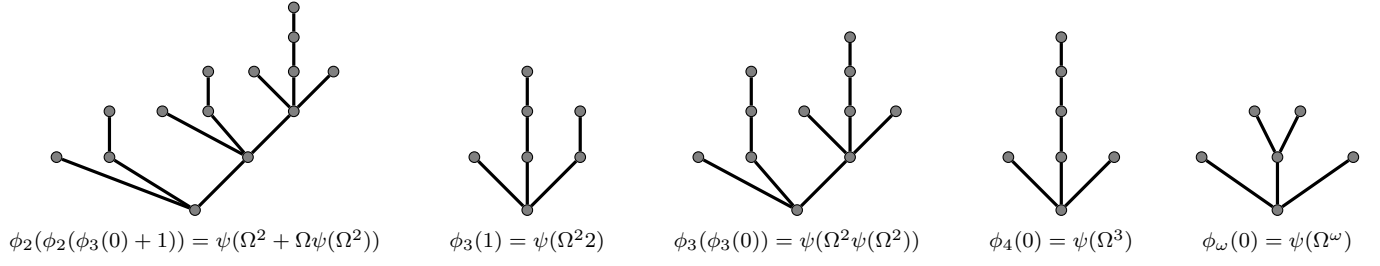
$$\phi_1(\phi_1(\phi_3(0) + 1)) = \psi(\Omega^2 + \psi(\Omega^2 + 1))$$

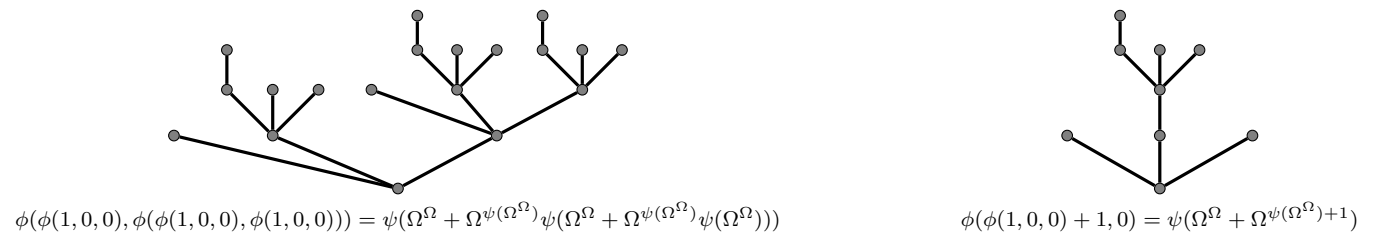
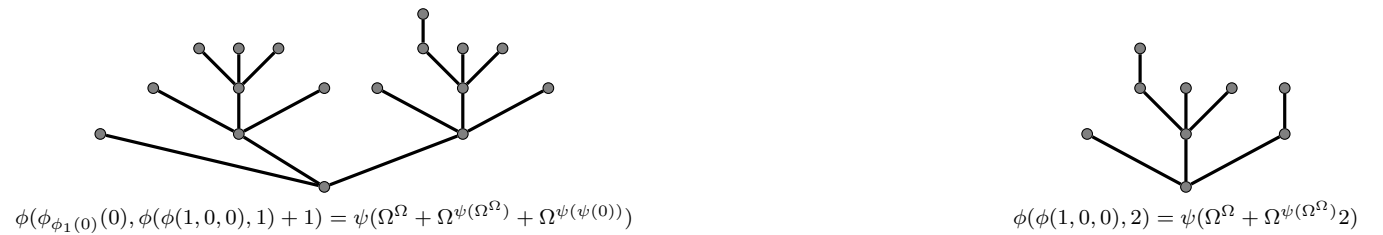
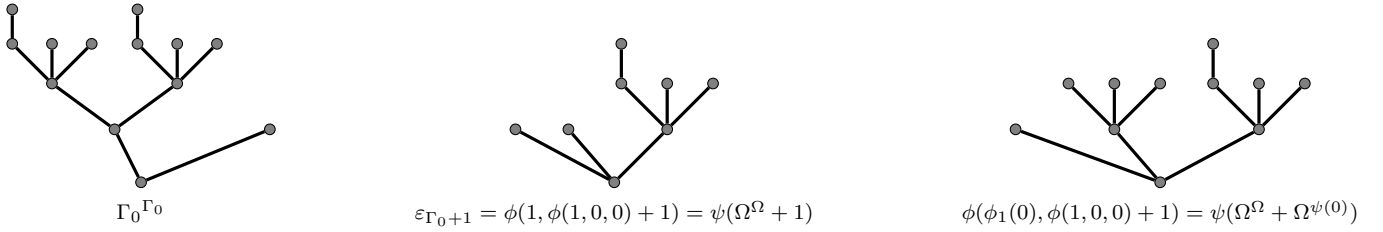
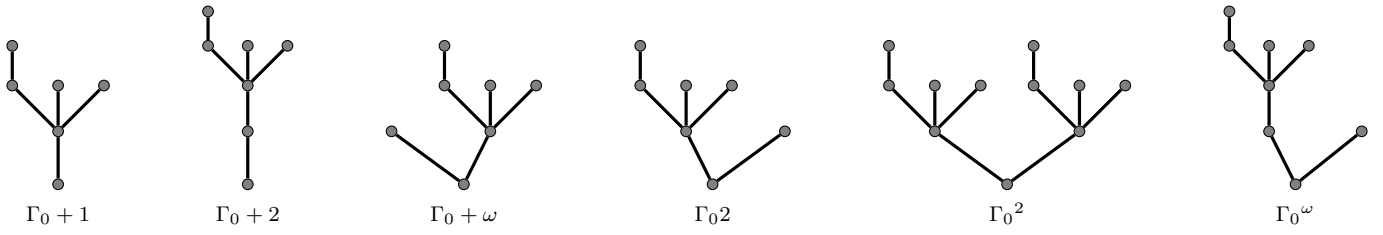


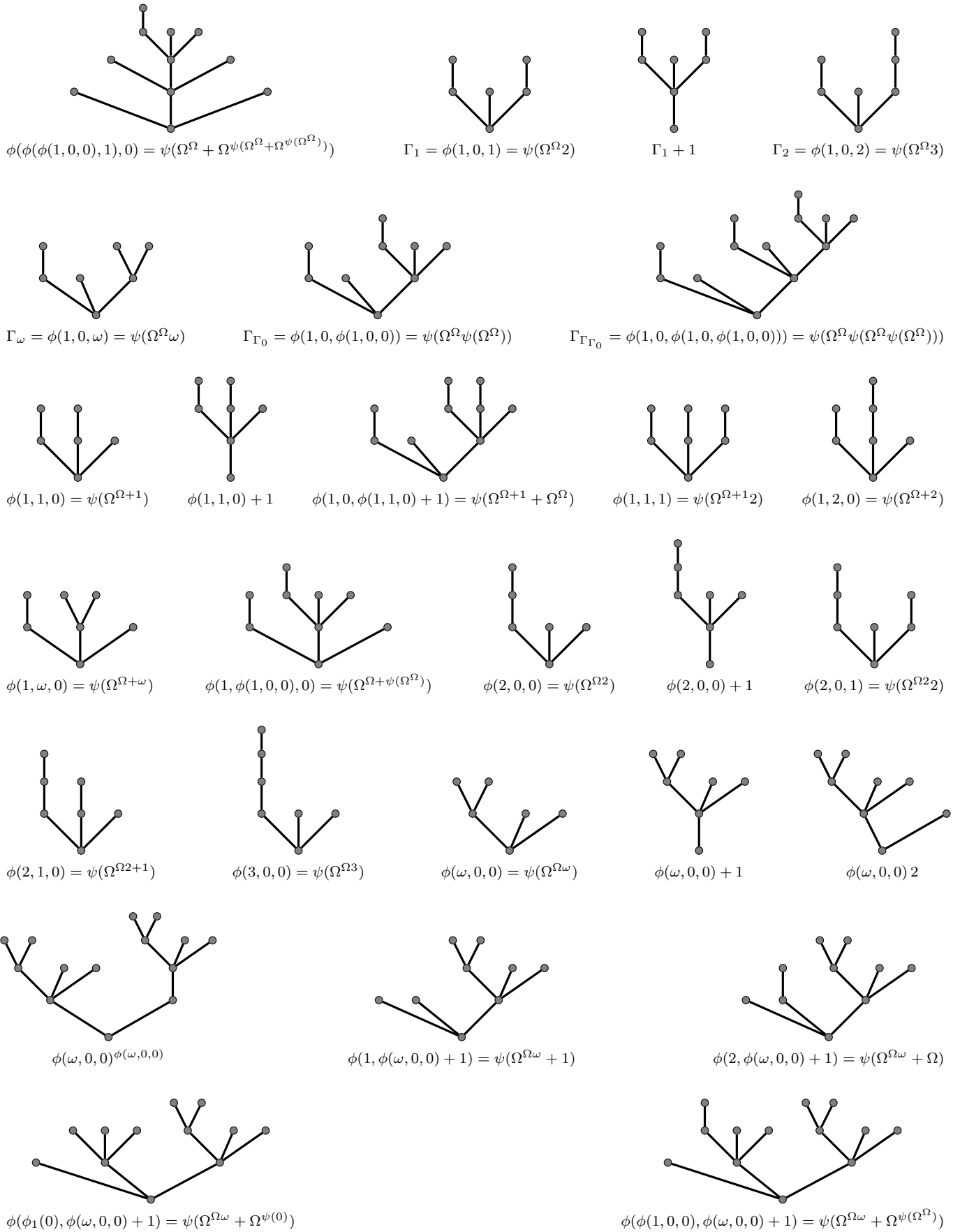
$$\phi_2(\phi_3(0) + 1) = \psi(\Omega^2 + \Omega)$$

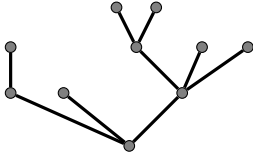


$$\phi_1(\phi_2(\phi_3(0) + 1) + 1) = \psi(\Omega^2 + \Omega + 1)$$

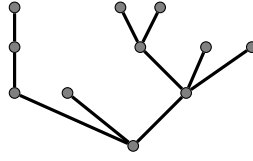




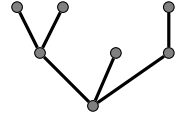




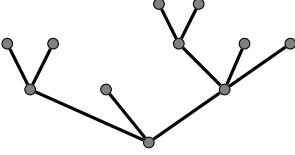
$$\phi(1, 0, \phi(\omega, 0, 0) + 1) = \psi(\Omega^{\Omega\omega} + \Omega^{\Omega^2})$$



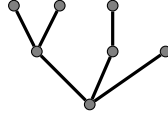
$$\phi(2, 0, \phi(\omega, 0, 0) + 1) = \psi(\Omega^{\Omega\omega} + \Omega^{\Omega^2})$$



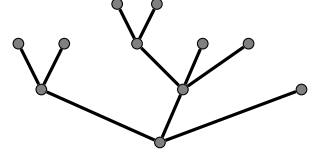
$$\phi(\omega, 0, 1) = \psi(\Omega^{\Omega\omega^2})$$



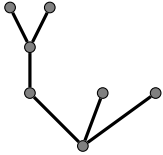
$$\phi(\omega, 0, \phi(\omega, 0, 0)) = \psi(\Omega^{\Omega\omega} \psi(\Omega^{\Omega\omega}))$$



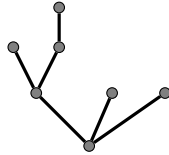
$$\phi(\omega, 1, 0) = \psi(\Omega^{\Omega\omega+1})$$



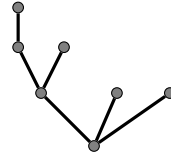
$$\phi(\omega, \phi(\omega, 0, 0), 0) = \psi(\Omega^{\Omega\omega} + \psi(\Omega^{\Omega\omega}))$$



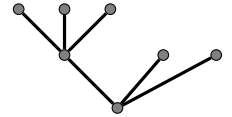
$$\phi(\omega + 1, 0, 0) = \psi(\Omega^{\Omega(\omega+1)})$$



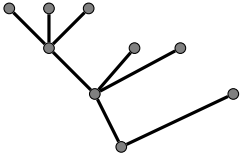
$$\phi(\omega^2, 0, 0) = \psi(\Omega^{\Omega\omega^2})$$



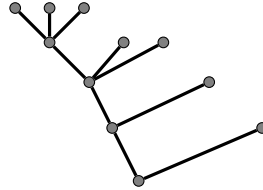
$$\phi(\omega^\omega, 0, 0) = \psi(\Omega^{\Omega\omega^\omega})$$



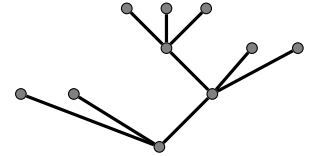
$$\phi(\phi_1(0), 0, 0) = \psi(\Omega^{\Omega\psi(0)})$$



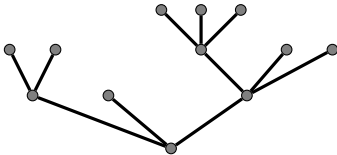
$$\phi(\phi_1(0), 0, 0) 2$$



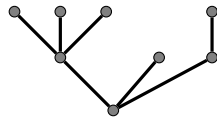
$$\phi(\phi_1(0), 0, 0) \phi(\phi_1(0), 0, 0)$$



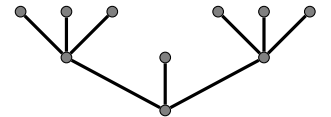
$$\phi(1, \phi(\phi_1(0), 0, 0) + 1) = \psi(\Omega^{\Omega\psi(0)} + 1)$$



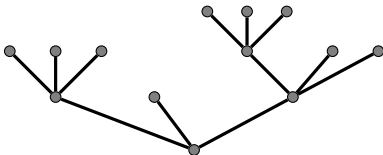
$$\phi(\omega, \phi(\phi_1(0), 0, 0) + 1) = \psi(\Omega^{\Omega\psi(0)} + \Omega^\omega)$$



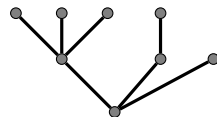
$$\phi(\phi_1(0), 0, 1) = \psi(\Omega^{\Omega\psi(0)} 2)$$



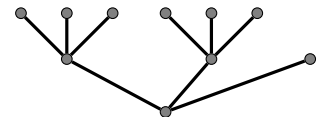
$$\phi(\phi_1(0), 0, \phi_1(0)) = \psi(\Omega^{\Omega\psi(0)} \psi(0))$$



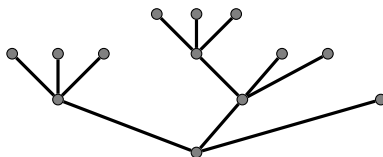
$$\phi(\phi_1(0), 0, \phi(\phi_1(0), 0, 0)) = \psi(\Omega^{\Omega\psi(0)} \psi(\Omega^{\Omega\psi(0)}))$$



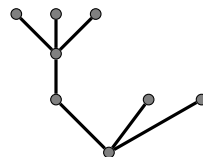
$$\phi(\phi_1(0), 1, 0) = \psi(\Omega^{\Omega\psi(0)+1})$$



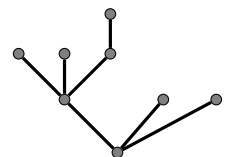
$$\phi(\phi_1(0), \phi_1(0), 0) = \psi(\Omega^{\Omega\psi(0)+\psi(0)})$$



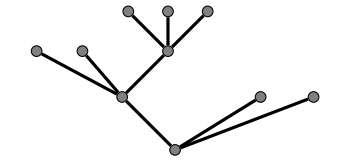
$$\phi(\phi_1(0), \phi(\phi_1(0), 0, 0), 0) = \psi(\Omega^{\Omega\psi(0)+\psi(\Omega^{\Omega\psi(0)})})$$



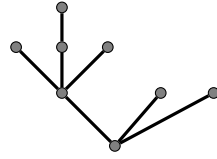
$$\phi(\phi_1(0) + 1, 0, 0) = \psi(\Omega^{\Omega(\psi(0)+1)})$$



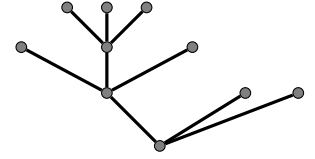
$$\phi(\phi_1(1), 0, 0) = \psi(\Omega^{\Omega\psi(1)})$$



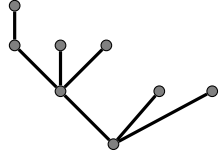
$$\phi(\phi_1(\phi_1(0)), 0, 0) = \psi(\Omega^{\Omega\psi(\psi(0))})$$



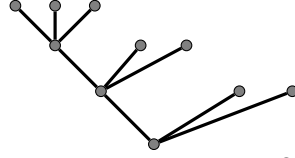
$$\phi(\phi_2(0), 0, 0) = \psi(\Omega^{\Omega\psi(\Omega)})$$



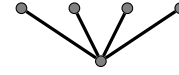
$$\phi(\phi_{\phi_1(0)}(0), 0, 0) = \psi(\Omega^{\Omega\psi(\Omega^{\psi(0)})})$$



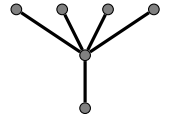
$$\phi(\phi(1, 0, 0), 0, 0) = \psi(\Omega^{\Omega\psi(\Omega^{\Omega})})$$



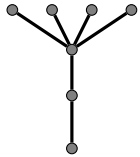
$$\phi(\phi(\phi_1(0), 0, 0), 0, 0) = \psi(\Omega^{\Omega\psi(\Omega^{\Omega\psi(0)})})$$



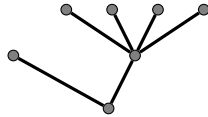
$$\phi(1, 0, 0, 0) = \psi(\Omega^{\Omega^2})$$



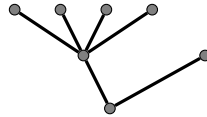
$$\phi(1, 0, 0, 0) + 1$$



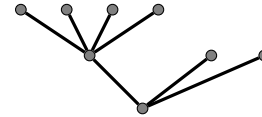
$$\phi(1, 0, 0, 0) + 2$$



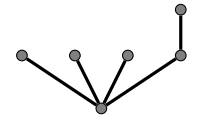
$$\phi(1, 0, 0, 0) + \omega$$



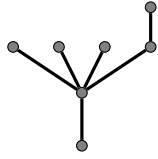
$$\phi(1, 0, 0, 0) \cdot 2$$



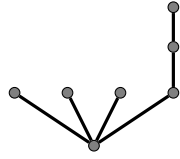
$$\phi(1, 0, \phi(1, 0, 0, 0) + 1) = \psi(\Omega^{\Omega^2 + \Omega^2})$$



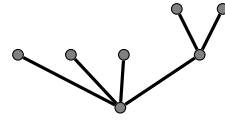
$$\phi(1, 0, 0, 1) = \psi(\Omega^{\Omega^2 \cdot 2})$$



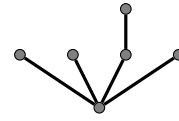
$$\phi(1, 0, 0, 1) + 1$$



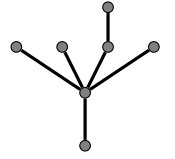
$$\phi(1, 0, 0, 2) = \psi(\Omega^{\Omega^2 \cdot 3})$$



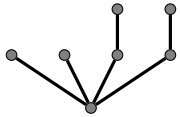
$$\phi(1, 0, 0, \omega) = \psi(\Omega^{\Omega^2 \cdot \omega})$$



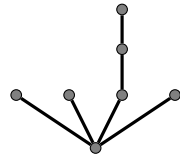
$$\phi(1, 0, 1, 0) = \psi(\Omega^{\Omega^2 + 1})$$



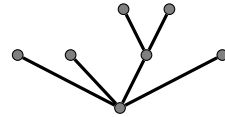
$$\phi(1, 0, 1, 0) + 1$$



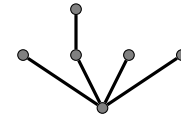
$$\phi(1, 0, 1, 1) = \psi(\Omega^{\Omega^2 + 1 \cdot 2})$$



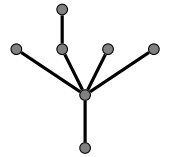
$$\phi(1, 0, 2, 0) = \psi(\Omega^{\Omega^2 + 2})$$



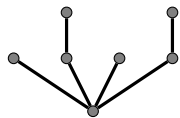
$$\phi(1, 0, \omega, 0) = \psi(\Omega^{\Omega^2 + \omega})$$



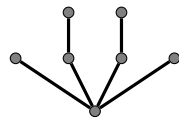
$$\phi(1, 1, 0, 0) = \psi(\Omega^{\Omega^2 + \Omega})$$



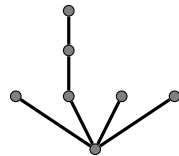
$$\phi(1, 1, 0, 0) + 1$$



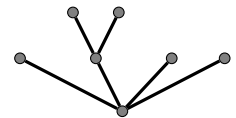
$$\phi(1, 1, 0, 1) = \psi(\Omega^{\Omega^2 + \Omega \cdot 2})$$



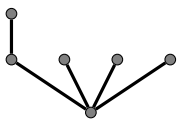
$$\phi(1, 1, 1, 0) = \psi(\Omega^{\Omega^2 + \Omega + 1})$$



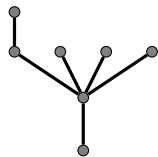
$$\phi(1, 2, 0, 0) = \psi(\Omega^{\Omega^2 + \Omega \cdot 2})$$



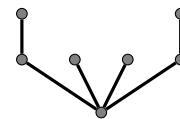
$$\phi(1, \omega, 0, 0) = \psi(\Omega^{\Omega^2 + \Omega \cdot \omega})$$



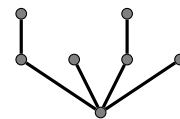
$$\phi(2, 0, 0, 0) = \psi(\Omega^{\Omega^2 \cdot 2})$$



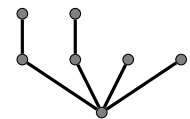
$$\phi(2, 0, 0, 0) + 1$$



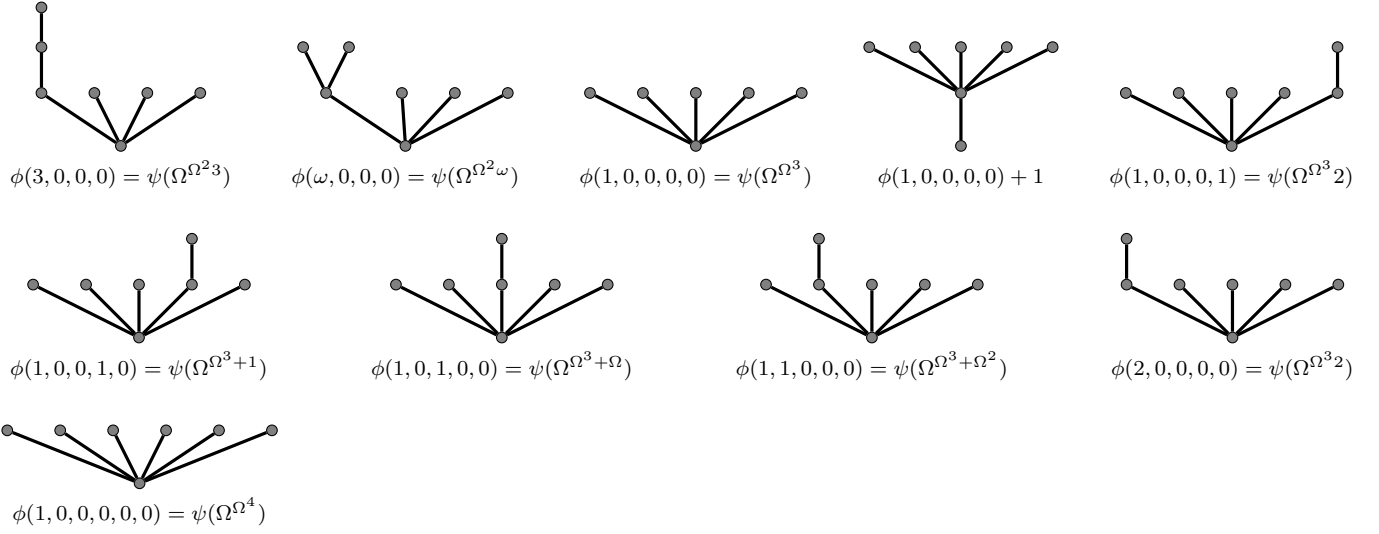
$$\phi(2, 0, 0, 1) = \psi(\Omega^{\Omega^2 \cdot 2 \cdot 2})$$



$$\phi(2, 0, 1, 0) = \psi(\Omega^{\Omega^2 \cdot 2 + 1})$$



$$\phi(2, 1, 0, 0) = \psi(\Omega^{\Omega^2 \cdot 2 + \Omega})$$



### Conventions and notations:

- The order on finite rooted trees is recursively defined as follows: a tree  $A$  is less than a tree  $B$ , written  $A \prec B$ , iff:
  - either there is some child (=immediate subtree)  $B'$  of  $B$  such that  $A \preceq B'$ ,
  - or the following two conditions hold: every child  $A'$  of  $A$  satisfies  $A' \prec B$ , and the list of children of  $A$  is lexicographically less than the list of children of  $B$  for the order  $\prec$  (with the leftmost children having the most weight, i.e., either  $B$  has more children than  $A$ , or if  $A'$  and  $B'$  are the leftmost children on which they differ then  $A' \prec B'$ ).

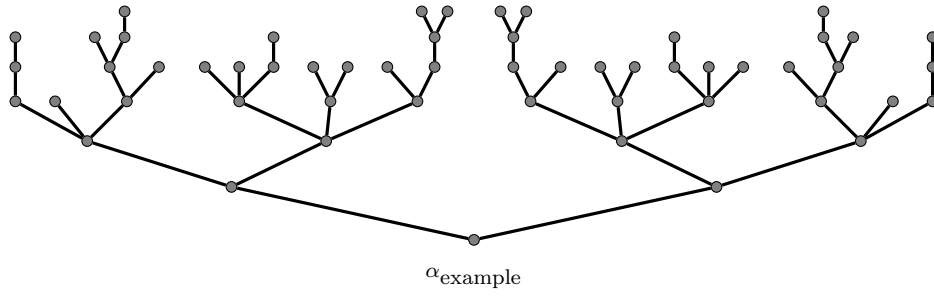
This is a well-order.

- $\omega$  is the smallest infinite ordinal.
- Ordinal addition, multiplication and exponentiation are defined as usual:
  - $\alpha + 0 = \alpha$
  - $\alpha + 1$  is the successor of  $\alpha$ .
  - $\alpha + (\beta + 1) = (\alpha + \beta) + 1$
  - If  $\delta$  is limit then  $\alpha + \delta$  is the limit of the  $\alpha + \beta$  for  $\beta < \delta$ .
  - $\alpha \cdot 0 = 0$
  - $\alpha(\beta + 1) = \alpha\beta + \alpha$
  - $\alpha^0 = 1$
  - $\alpha^{\beta+1} = \alpha^\beta \alpha$
- $\phi_1(\alpha) = \varepsilon_\alpha$  is the  $\alpha$ -th fixed point of  $\xi \mapsto \omega^\xi$  (with  $\varepsilon_0$  the smallest: it is the limit of  $\omega, \omega^\omega, \omega^{\omega^\omega}, \dots$ ).
- *The Veblen hierarchy:* For any  $\beta > 1$ , let  $\phi_\beta(\alpha)$  be the  $\alpha$ -th common fixed point of  $\phi_\gamma$  for all  $0 < \gamma < \beta$  (with  $\phi_\beta(0)$  the smallest). Note:
  - $\phi_\beta(\alpha)$  is continuous in  $\alpha$  for all  $\beta$  but it is *not* continuous in  $\beta$  except for  $\alpha = 0$ .
  - $\phi_{\beta+1}(0)$  is the limit of  $\phi_\beta(0), \phi_\beta(\phi_\beta(0)), \phi_\beta(\phi_\beta(\phi_\beta(0))), \dots$ .
  - $\phi_{\beta+1}(\alpha + 1)$  is the limit of  $\phi_{\beta+1}(\alpha) + 1, \phi_\beta(\phi_{\beta+1}(\alpha) + 1), \phi_\beta(\phi_\beta(\phi_{\beta+1}(\alpha) + 1)), \dots$ .
  - If  $\delta$  is limit (and  $\beta$  is arbitrary) then  $\phi_\beta(\delta)$  is the limit of the  $\phi_\beta(\xi)$  for  $\xi < \delta$ .
  - If  $\delta$  is limit then  $\phi_\delta(0)$  is the limit of the  $\phi_\gamma(0)$  for  $\gamma < \delta$ .
  - If  $\delta$  is limit then  $\phi_\delta(\alpha + 1)$  is the limit of the  $\phi_\gamma(\phi_\delta(\alpha) + 1)$  for  $\gamma < \delta$ .

- $\phi(1, 0, \alpha) = \Gamma_\alpha$  is the  $\alpha$ -th fixed point of  $\xi \mapsto \phi_\xi(0)$  (with  $\Gamma_0$  the smallest: it is the limit of  $\phi_1(0)$ ,  $\phi_{\phi_1(0)}(0)$ ,  $\phi_{\phi_{\phi_1(0)}(0)}(0)$ , etc.; this is known as the Feferman-Schütte ordinal).
- More generally, let  $\phi(\beta, \alpha) = \phi_\beta(\alpha)$  (we only use the notation with a subscript for values less than  $\Gamma_0$ ) and define  $\phi$  of finitely many variables by:  $\phi(\beta_n, \beta_{n-1}, \dots, \beta_r, 0, 0, \dots, 0, \alpha)$  is the  $\alpha$ -th fixed point of all the  $\xi \mapsto \phi(\gamma_n, \gamma_{n-1}, \dots, \gamma_r, \xi, 0, 0, \dots, 0)$  for  $(\gamma_n, \dots, \gamma_r)$  lexicographically less than  $(\beta_n, \dots, \beta_r)$  (the leftmost variable having the most weight), with the convention that  $\phi(0, \beta_{n-1}, \dots, \beta_0) = \phi(\beta_{n-1}, \dots, \beta_0)$ .
- $\Omega$  is the first uncountable ordinal.
- A *collapsing function*:  $\psi(\alpha)$  is defined inductively as the smallest ordinal not expressible from 0, 1,  $\omega$  and  $\Omega$  using addition, multiplication, exponentiation, and application of the function  $\psi$  itself to ordinals less than  $\alpha$ . Or, more rigorously:
  - Assume  $\psi$  has been defined for all ordinals  $\beta < \alpha$ .
  - Let  $C(\alpha)$  be the set of ordinals constructed starting from 0, 1,  $\omega$  and  $\Omega$  by recursively applying the following functions: ordinal addition, multiplication and exponentiation and the function  $\psi|_\alpha$ , i.e., the restriction of  $\psi$  to ordinals  $\beta < \alpha$ . Formally, we define  $C(\alpha)_0 = \{0, 1, \omega, \Omega\}$  and inductively  $C(\alpha)_{n+1} = C(\alpha)_n \cup \{\beta_1 + \beta_2, \beta_1\beta_2, \beta_1^{\beta_2} : \beta_1, \beta_2 \in C(\alpha)_n\} \cup \{\psi(\beta) : \beta \in C(\alpha)_n \wedge \beta < \alpha\}$  for all natural numbers  $n$ , and we let  $C(\alpha)$  be the union of the  $C(\alpha)_n$  for all  $n$ .
  - Define  $\psi(\alpha)$  as the smallest ordinal not in  $C(\alpha)$ .

It turns out that:

- $\psi$  is continuous and non-decreasing.
  - The range of  $\psi$  is precisely the set of  $\varepsilon$ -numbers (i.e., fixed points of  $\xi \mapsto \omega^\xi$ ) up to the Bachmann-Howard ordinal.
  - $C(\alpha)$  consists exactly of those ordinals whose iterated base  $\Omega$  representation only has pieces less than  $\psi(\alpha)$ ; in particular,  $C(\alpha) \cap \Omega = \alpha$ .
  - $\psi(\alpha + 1)$  is always equal either to the first  $\varepsilon$ -number after  $\psi(\alpha)$ —this happens when  $\alpha \in C(\alpha)$ —or else to  $\psi(\alpha)$ —which happens when  $\alpha \notin C(\alpha)$ .
  - If  $\delta$  is limit then  $\psi(\delta)$  is the limit of the  $\psi(\xi)$  for  $\xi < \delta$ .
  - $\psi(\alpha + \Omega)$  is the first fixed point of  $\xi \mapsto \psi(\alpha + \xi)$ .
  - $\psi(\alpha\Omega)$  is the first fixed point of  $\xi \mapsto \psi(\alpha\xi)$ .
  - $\psi(\alpha^\Omega)$  is the first fixed point of  $\xi \mapsto \psi(\alpha^\xi)$ .
  - $\psi(\alpha^{\beta^\Omega})$  is the first fixed point of  $\xi \mapsto \psi(\alpha^{\beta^\xi})$ .
- Certain ordinals have special names:
    - $\psi(\Omega^\Omega) = \phi(1, 0, 0) = \Gamma_0$  is the “Feferman-Schütte ordinal”: it is the set of ordinals constructed starting from 0, 1,  $\omega$  by recursively applying ordinal addition, multiplication and exponentiation and the Veblen functions  $\phi_\beta(\alpha)$  of two variables;
    - $\psi(\Omega^{\Omega^\omega})$  is the “small” Veblen ordinal: it is the set of ordinals constructed starting from 0, 1,  $\omega$  by recursively applying ordinal addition, multiplication and exponentiation and the Veblen functions  $\phi(\dots)$  of finitely many variables, and it is also the length of the ordering  $<$  of finite rooted trees;
    - $\psi(\Omega^{\Omega^\Omega})$  is the “large” Veblen ordinal (it can be defined similarly to the “small” Veblen ordinal using a generalization of  $\phi(\dots)$  to transfinitely many variables all but finitely many of which are zero);
    - $\psi(\varepsilon_{\Omega+1})$  is the Bachmann-Howard ordinal.



$$\begin{aligned}
 \alpha_{\text{example}} &= \phi(\phi_1(1), \omega, \omega^2 + \omega) \phi(2, 0, \omega^{\omega^2}) \phi(\omega^{\omega+1}, \omega, \phi(1, 0, 0)) \phi(\omega^\omega + \omega, 0, 2) \\
 &= \psi(\Omega^{\Omega\psi(1)+\omega}(\omega^2 + \omega)) \psi(\Omega^{\Omega^2\omega^{\omega^2}}) \psi(\Omega^{\Omega\omega^{\omega+1}+\omega} \psi(\Omega^\Omega)) \psi(\Omega^{\Omega(\omega^\omega+\omega)} 3)
 \end{aligned}$$