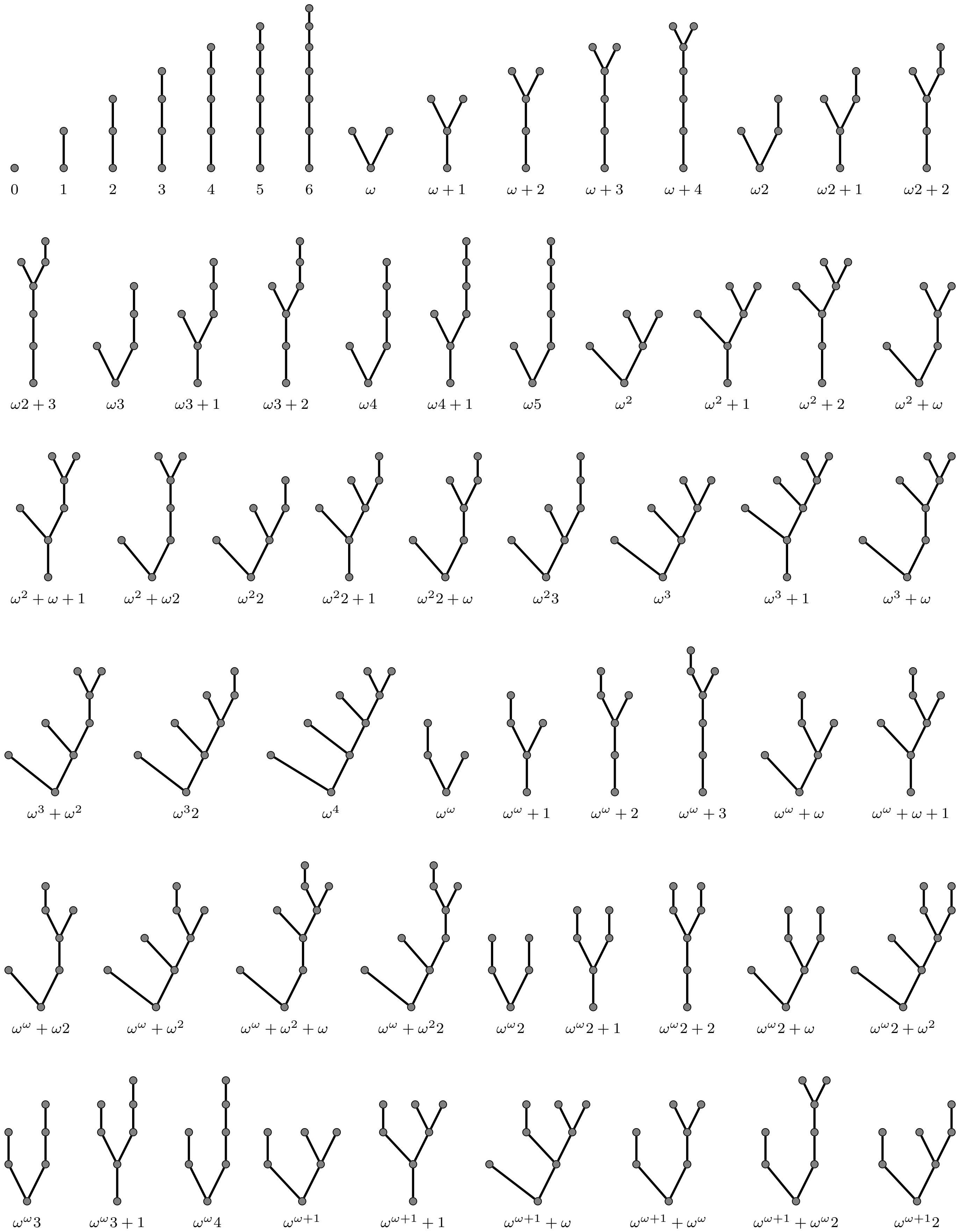
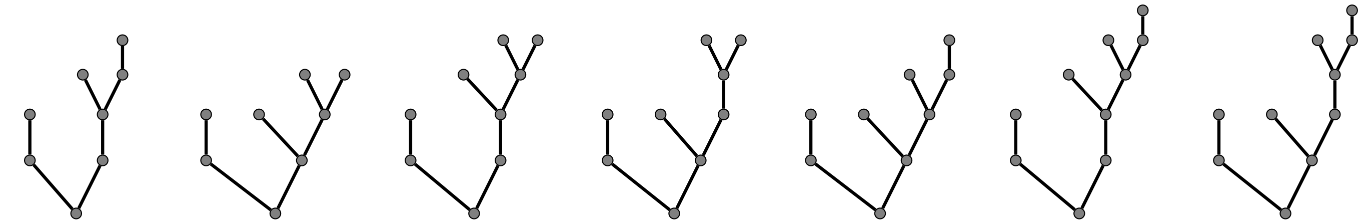
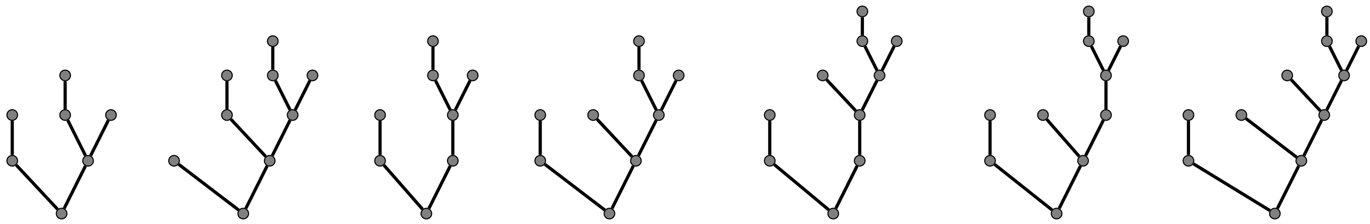
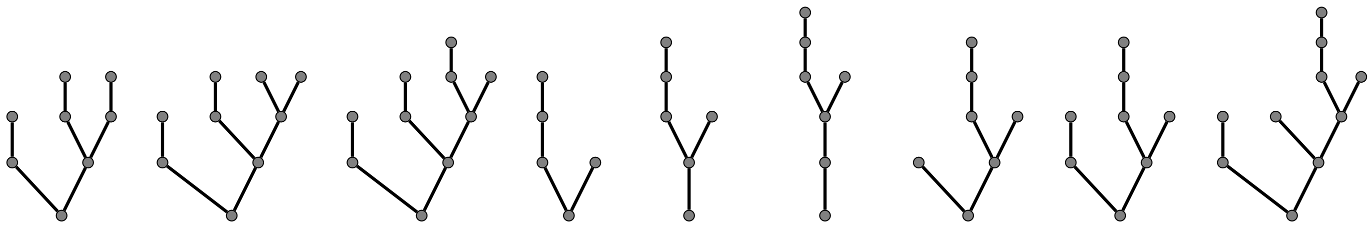

0 1 2 3 4 5 6 ω $\omega+1$ $\omega+2$ $\omega+3$ $\omega+4$ $\omega2$ $\omega2+1$ $\omega2+2$
$\omega2+3$ $\omega3$ $\omega3+1$ $\omega3+2$ $\omega4$ $\omega4+1$ $\omega5$ ω^2 ω^2+1 ω^2+2 $\omega^2+\omega$
$\omega^2+\omega+1$ $\omega^2+\omega2$ $\omega^2 2$ $\omega^2 2+1$ $\omega^2 2+\omega$ $\omega^2 3$ ω^3 ω^3+1 $\omega^3+\omega$
$\omega^3+\omega^2$ $\omega^3 2$ ω^4 ω^ω $\omega^\omega+1$ $\omega^\omega+2$ $\omega^\omega+3$ $\omega^\omega+\omega$ $\omega^\omega+\omega+1$
$\omega^\omega+\omega2$ $\omega^\omega+\omega^2$ $\omega^\omega+\omega^2+\omega$ $\omega^\omega+\omega^2 2$ $\omega^\omega 2$ $\omega^\omega 2+1$ $\omega^\omega 2+2$ $\omega^\omega 2+\omega$ $\omega^\omega 2+\omega^2$
$\omega^\omega 3$ $\omega^\omega 3+1$ $\omega^\omega 4$ $\omega^{\omega+1}$ $\omega^{\omega+1}+1$ $\omega^{\omega+1}+\omega$ $\omega^{\omega+1}+\omega^\omega$ $\omega^{\omega+1}+\omega^\omega 2$ $\omega^{\omega+1}2$



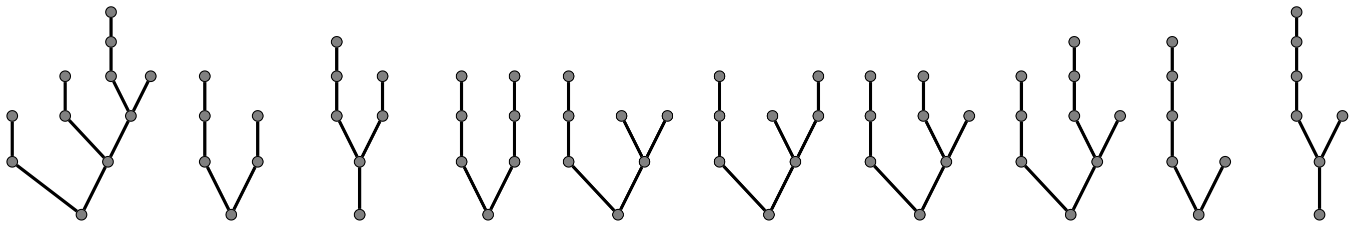
$\omega^{\omega+1}2+\omega^{\omega}$ $\omega^{\omega+2}$ $\omega^{\omega+2}+\omega^{\omega}$ $\omega^{\omega+2}+\omega^{\omega+1}$ $\omega^{\omega+2}2$ $\omega^{\omega+2}2+\omega^{\omega}$ $\omega^{\omega+2}2+\omega^{\omega+1}$



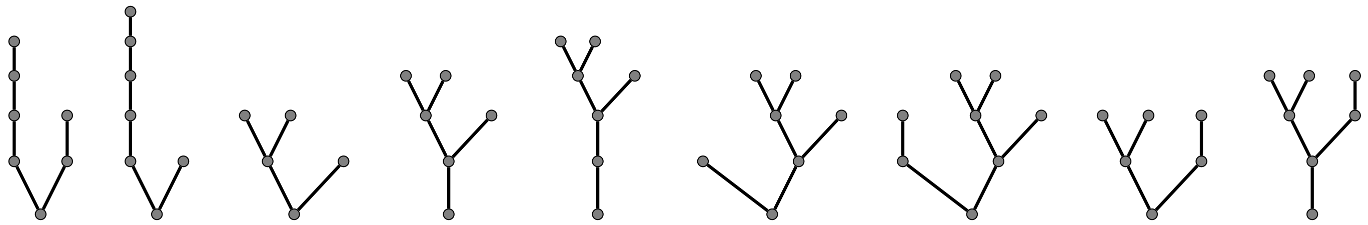
$\omega^{\omega 2}$ $\omega^{\omega 2}+\omega$ $\omega^{\omega 2}+\omega^{\omega}$ $\omega^{\omega 2}+\omega^{\omega+1}$ $\omega^{\omega 2}+\omega^{\omega+1}+\omega^{\omega}$ $\omega^{\omega 2}+\omega^{\omega+1}2$ $\omega^{\omega 2}+\omega^{\omega+2}$



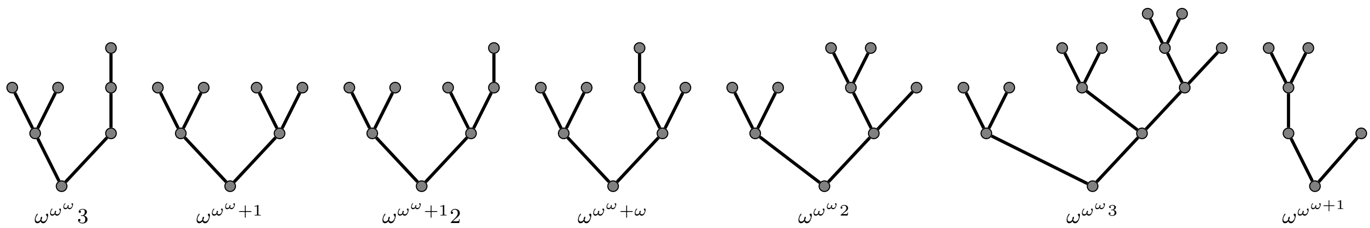
$\omega^{\omega 2}2$ $\omega^{\omega 2+1}$ $\omega^{\omega 3}$ ω^{ω^2} $\omega^{\omega^2}+1$ $\omega^{\omega^2}+2$ $\omega^{\omega^2}+\omega$ $\omega^{\omega^2}+\omega^{\omega}$ $\omega^{\omega^2}+\omega^{\omega+1}$



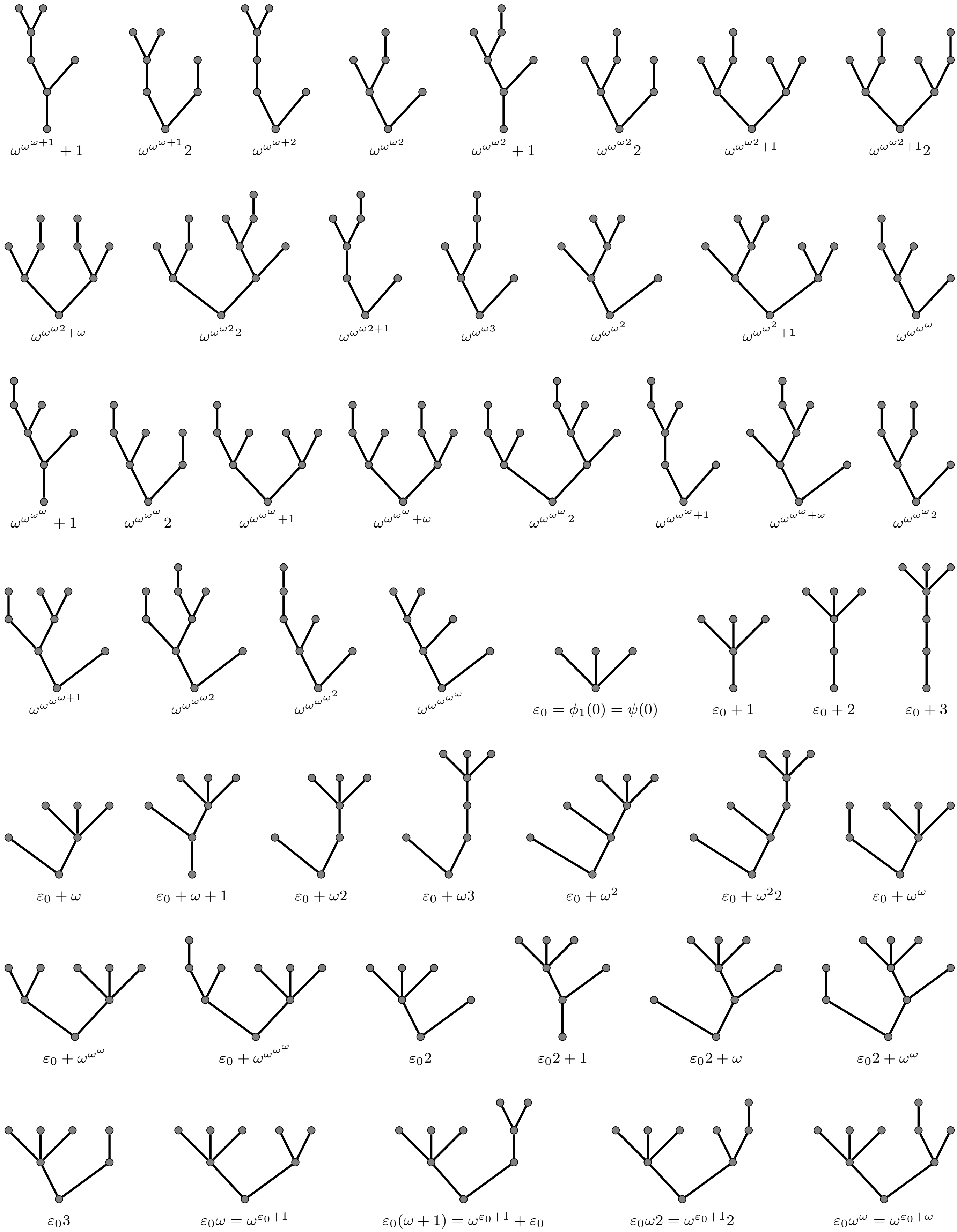
$\omega^{\omega^2}+\omega^{\omega 2}$ $\omega^{\omega^2}2$ $\omega^{\omega^2}2+1$ $\omega^{\omega^2}3$ ω^{ω^2+1} $\omega^{\omega^2+1}2$ $\omega^{\omega^2+\omega}$ $\omega^{\omega^2 2}$ ω^{ω^3} $\omega^{\omega^3}+1$



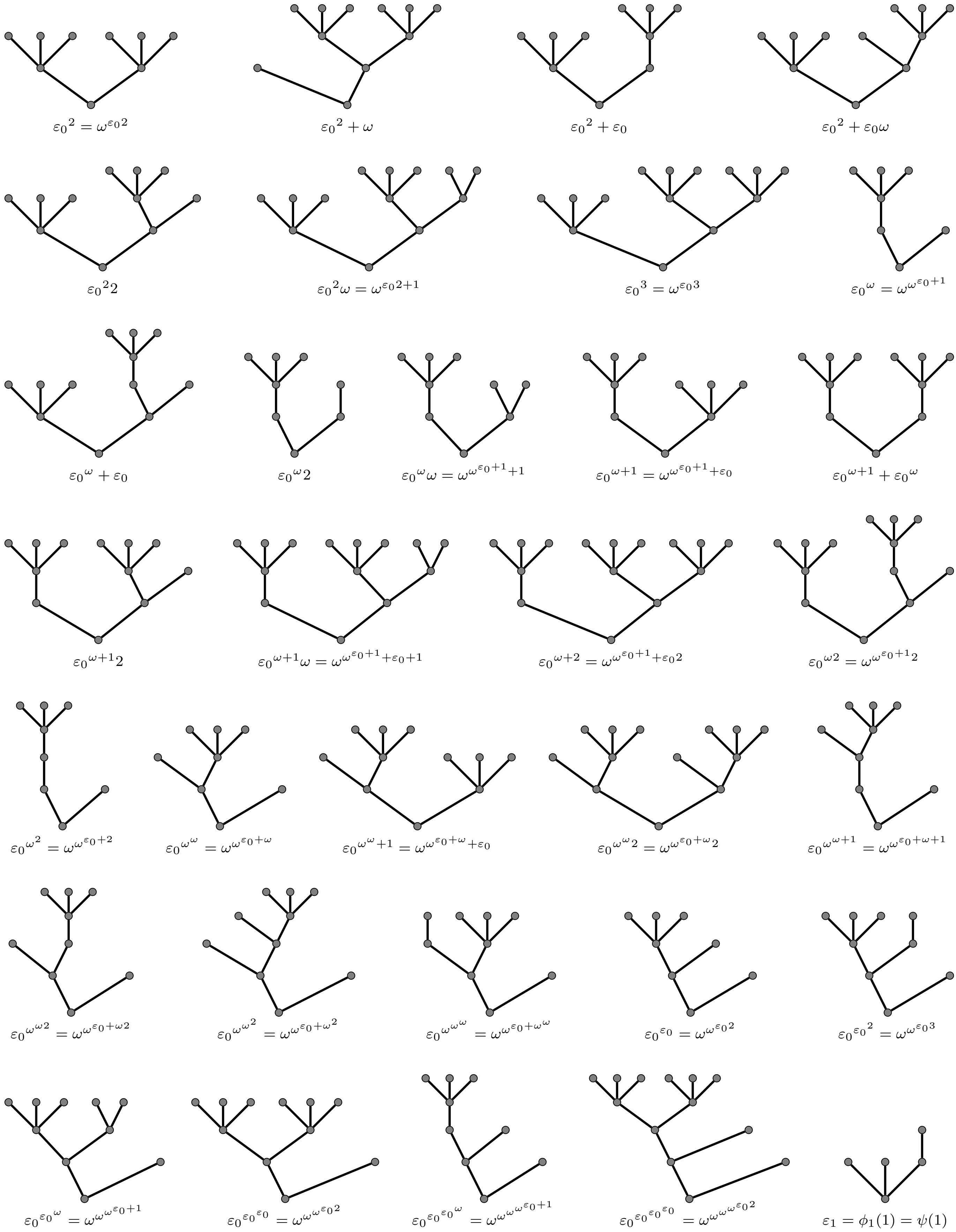
$\omega^{\omega^3}2$ ω^{ω^4} $\omega^{\omega^{\omega}}$ $\omega^{\omega^{\omega}}+1$ $\omega^{\omega^{\omega}}+2$ $\omega^{\omega^{\omega}}+\omega$ $\omega^{\omega^{\omega}}+\omega^{\omega}$ $\omega^{\omega^{\omega}}2$ $\omega^{\omega^{\omega}}2+1$



$\omega^{\omega^{\omega}}3$ $\omega^{\omega^{\omega}+1}$ $\omega^{\omega^{\omega}+1}2$ $\omega^{\omega^{\omega}+\omega}$ $\omega^{\omega^{\omega}2}$ $\omega^{\omega^{\omega}3}$ $\omega^{\omega^{\omega+1}}$



$\omega^{\omega^{\omega+1}}+1$
$\omega^{\omega^{\omega+1}}2$
$\omega^{\omega^{\omega+2}}$
$\omega^{\omega^{\omega 2}}$
$\omega^{\omega^{\omega 2}}+1$
$\omega^{\omega^{\omega 2}}2$
$\omega^{\omega^{\omega 2}+1}$
$\omega^{\omega^{\omega 2}+1}2$
$\omega^{\omega^{\omega 2}+\omega}$
$\omega^{\omega^{\omega 2}2}$
$\omega^{\omega^{\omega 2+1}}$
$\omega^{\omega^{\omega 3}}$
$\omega^{\omega^{\omega^2}}$
$\omega^{\omega^{\omega^2}+1}$
$\omega^{\omega^{\omega^\omega}}$
$\omega^{\omega^{\omega^\omega}}+1$
$\omega^{\omega^{\omega^\omega}}2$
$\omega^{\omega^{\omega^\omega}+1}$
$\omega^{\omega^{\omega^\omega}+\omega}$
$\omega^{\omega^{\omega^\omega}2}$
$\omega^{\omega^{\omega^\omega+1}}$
$\omega^{\omega^{\omega^\omega+\omega}}$
$\omega^{\omega^{\omega^\omega 2}}$
$\omega^{\omega^{\omega^{\omega+1}}}$
$\omega^{\omega^{\omega^{\omega 2}}}$
$\omega^{\omega^{\omega^{\omega^2}}}$
$\omega^{\omega^{\omega^{\omega^\omega}}}$
$\varepsilon_0=\phi_1(0)=\psi(0)$
ε_0+1
ε_0+2
ε_0+3
$\varepsilon_0+\omega$
$\varepsilon_0+\omega+1$
$\varepsilon_0+\omega 2$
$\varepsilon_0+\omega 3$
$\varepsilon_0+\omega^2$
$\varepsilon_0+\omega^2 2$
$\varepsilon_0+\omega^\omega$
$\varepsilon_0+\omega^{\omega^\omega}$
$\varepsilon_0+\omega^{\omega^{\omega^\omega}}$
$\varepsilon_0 2$
$\varepsilon_0 2+1$
$\varepsilon_0 2+\omega$
$\varepsilon_0 2+\omega^\omega$
$\varepsilon_0 3$
$\varepsilon_0\omega=\omega^{\varepsilon_0+1}$
$\varepsilon_0(\omega+1)=\omega^{\varepsilon_0+1}+\varepsilon_0$
$\varepsilon_0\omega 2=\omega^{\varepsilon_0+1}2$
$\varepsilon_0\omega^\omega=\omega^{\varepsilon_0+\omega}$



$\varepsilon_0{}^2 = \omega^{\varepsilon_0 2}$ | $\varepsilon_0{}^2 + \omega$ | $\varepsilon_0{}^2 + \varepsilon_0$ | $\varepsilon_0{}^2 + \varepsilon_0\omega$

$\varepsilon_0{}^2 2$ | $\varepsilon_0{}^2\omega = \omega^{\varepsilon_0 2+1}$ | $\varepsilon_0{}^3 = \omega^{\varepsilon_0 3}$ | $\varepsilon_0{}^\omega = \omega^{\omega^{\varepsilon_0+1}}$

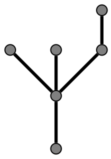
$\varepsilon_0{}^\omega + \varepsilon_0$ | $\varepsilon_0{}^\omega 2$ | $\varepsilon_0{}^\omega\omega = \omega^{\omega^{\varepsilon_0+1}+1}$ | $\varepsilon_0{}^{\omega+1} = \omega^{\omega^{\varepsilon_0+1}+\varepsilon_0}$ | $\varepsilon_0{}^{\omega+1} + \varepsilon_0{}^\omega$

$\varepsilon_0{}^{\omega+1}2$ | $\varepsilon_0{}^{\omega+1}\omega = \omega^{\omega^{\varepsilon_0+1}+\varepsilon_0+1}$ | $\varepsilon_0{}^{\omega+2} = \omega^{\omega^{\varepsilon_0+1}+\varepsilon_0 2}$ | $\varepsilon_0{}^{\omega 2} = \omega^{\omega^{\varepsilon_0+1}2}$

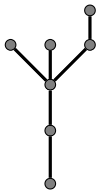
$\varepsilon_0{}^{\omega^2} = \omega^{\omega^{\varepsilon_0+2}}$ | $\varepsilon_0{}^{\omega^\omega} = \omega^{\omega^{\varepsilon_0+\omega}}$ | $\varepsilon_0{}^{\omega^\omega+1} = \omega^{\omega^{\varepsilon_0+\omega}+\varepsilon_0}$ | $\varepsilon_0{}^{\omega^\omega 2} = \omega^{\omega^{\varepsilon_0+\omega}2}$ | $\varepsilon_0{}^{\omega^{\omega+1}} = \omega^{\omega^{\varepsilon_0+\omega+1}}$

$\varepsilon_0{}^{\omega^{\omega 2}} = \omega^{\omega^{\varepsilon_0+\omega 2}}$ | $\varepsilon_0{}^{\omega^{\omega^2}} = \omega^{\omega^{\varepsilon_0+\omega^2}}$ | $\varepsilon_0{}^{\omega^{\omega^\omega}} = \omega^{\omega^{\varepsilon_0+\omega^\omega}}$ | $\varepsilon_0{}^{\varepsilon_0} = \omega^{\omega^{\varepsilon_0 2}}$ | $\varepsilon_0{}^{\varepsilon_0{}^2} = \omega^{\omega^{\varepsilon_0 3}}$

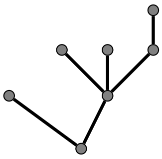
$\varepsilon_0{}^{\varepsilon_0{}^\omega} = \omega^{\omega^{\omega^{\varepsilon_0+1}}}$ | $\varepsilon_0{}^{\varepsilon_0{}^{\varepsilon_0}} = \omega^{\omega^{\omega^{\varepsilon_0 2}}}$ | $\varepsilon_0{}^{\varepsilon_0{}^{\varepsilon_0{}^\omega}} = \omega^{\omega^{\omega^{\omega^{\varepsilon_0+1}}}}$ | $\varepsilon_0{}^{\varepsilon_0{}^{\varepsilon_0{}^{\varepsilon_0}}} = \omega^{\omega^{\omega^{\omega^{\varepsilon_0 2}}}}$ | $\varepsilon_1 = \phi_1(1) = \psi(1)$



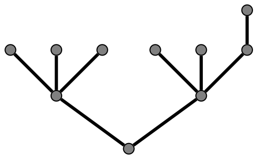
$\varepsilon_1 + 1$



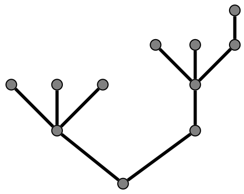
$\varepsilon_1 + 2$



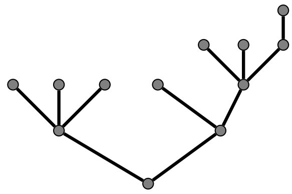
$\varepsilon_1 + \omega$



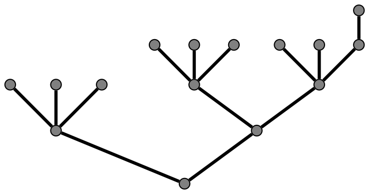
$\varepsilon_1 + \varepsilon_0$



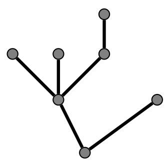
$\varepsilon_1 + \varepsilon_0 2$



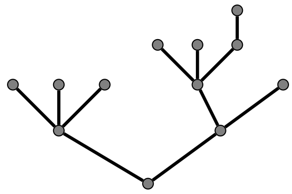
$\varepsilon_1 + \varepsilon_0 \omega$



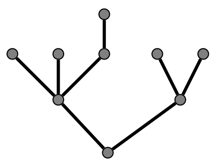
$\varepsilon_1 + {\varepsilon_0}^2$



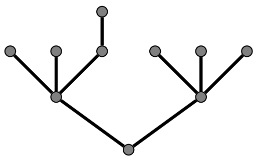
$\varepsilon_1 2$



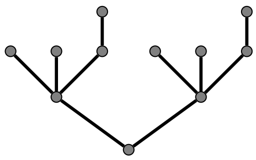
$\varepsilon_1 2 + \varepsilon_0$



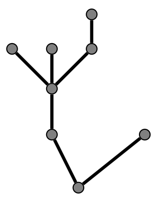
$\varepsilon_1 \omega = \omega^{\varepsilon_1 + 1}$



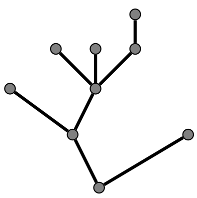
$\varepsilon_1 \varepsilon_0 = \omega^{\varepsilon_1 + \varepsilon_0}$



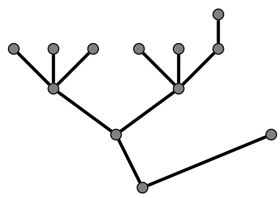
${\varepsilon_1}^2 = \omega^{\varepsilon_1 2}$



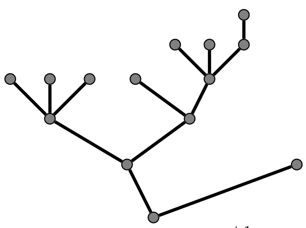
${\varepsilon_1}^{\omega} = \omega^{\omega^{\varepsilon_1 + 1}}$



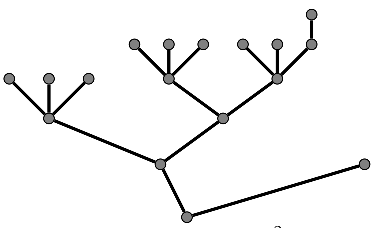
${\varepsilon_1}^{\omega^{\omega}} = \omega^{\omega^{\varepsilon_1 + \omega}}$



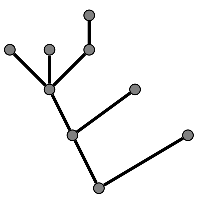
${\varepsilon_1}^{\varepsilon_0} = \omega^{\omega^{\varepsilon_1 + \varepsilon_0}}$



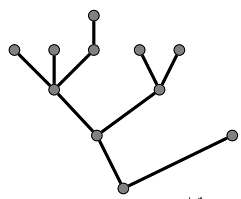
${\varepsilon_1}^{{\varepsilon_0}^{\omega}} = \omega^{\omega^{\varepsilon_1 + \omega^{\varepsilon_0 + 1}}}$



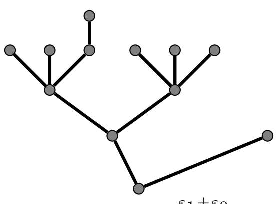
${\varepsilon_1}^{{\varepsilon_0}^{\varepsilon_0}} = \omega^{\omega^{\varepsilon_1 + \omega^{\varepsilon_0 2}}}$



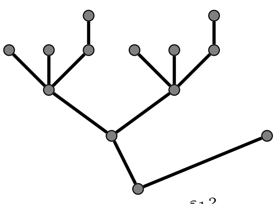
${\varepsilon_1}^{\varepsilon_1} = \omega^{\omega^{\varepsilon_1 2}}$



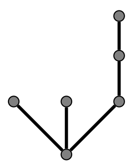
${\varepsilon_1}^{{\varepsilon_1}^{\omega}} = \omega^{\omega^{\omega^{\varepsilon_1 + 1}}}$



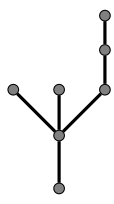
${\varepsilon_1}^{{\varepsilon_1}^{\varepsilon_0}} = \omega^{\omega^{\omega^{\varepsilon_1 + \varepsilon_0}}}$



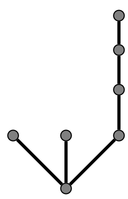
${\varepsilon_1}^{{\varepsilon_1}^{\varepsilon_1}} = \omega^{\omega^{\omega^{\varepsilon_1 2}}}$



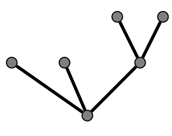
$\varepsilon_2 = \phi_1(2) = \psi(2)$



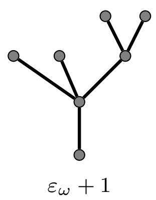
$\varepsilon_2 + 1$



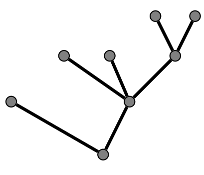
$\varepsilon_3 = \phi_1(3) = \psi(3)$



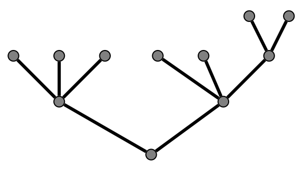
$\varepsilon_\omega = \phi_1(\omega) = \psi(\omega)$



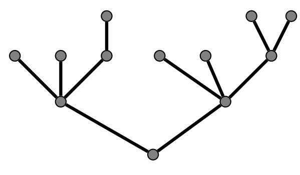
$\varepsilon_\omega + 1$



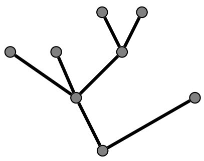
$\varepsilon_\omega + \omega$



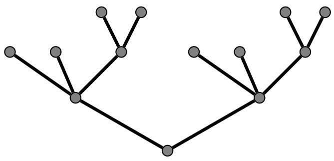
$\varepsilon_\omega + \varepsilon_0$



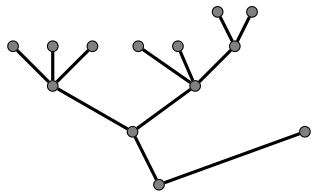
$\varepsilon_\omega + \varepsilon_1$



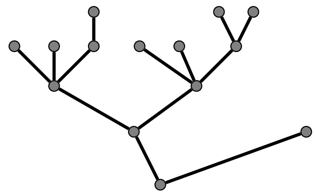
$\varepsilon_\omega 2$



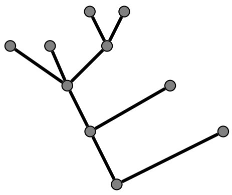
$\varepsilon_\omega{}^2$



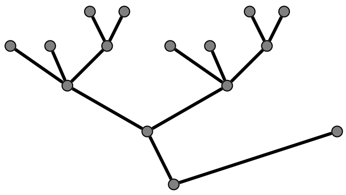
$\varepsilon_\omega{}^{\varepsilon_0} = \phi_1(\omega)^{\phi_1(0)}$



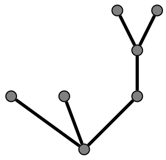
$\varepsilon_\omega{}^{\varepsilon_1} = \phi_1(\omega)^{\phi_1(1)}$



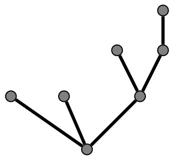
$\varepsilon_\omega{}^{\varepsilon_\omega} = \phi_1(\omega)^{\phi_1(\omega)}$



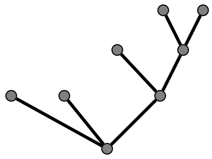
$\varepsilon_\omega{}^{\varepsilon_\omega{}^{\varepsilon_\omega}} = \phi_1(\omega)^{\phi_1(\omega)^{\phi_1(\omega)}}$



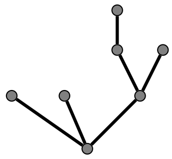
$\varepsilon_{\omega+1} = \phi_1(\omega+1) = \psi(\omega+1)$



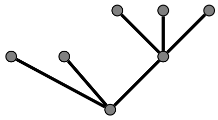
$\varepsilon_{\omega 2} = \phi_1(\omega 2) = \psi(\omega 2)$



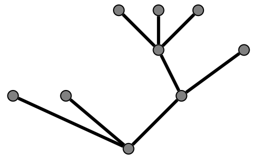
$\varepsilon_{\omega^2} = \phi_1(\omega^2) = \psi(\omega^2)$



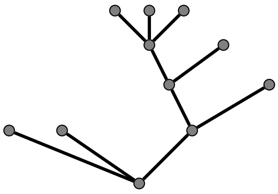
$\varepsilon_{\omega^\omega} = \phi_1(\omega^\omega) = \psi(\omega^\omega)$



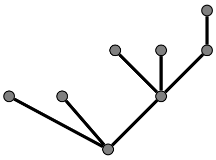
$\varepsilon_{\varepsilon_0} = \phi_1(\phi_1(0)) = \psi(\psi(0))$



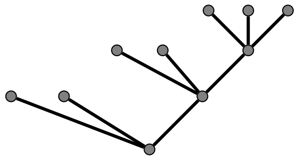
$\varepsilon_{\varepsilon_0 2} = \phi_1(\phi_1(0)\,2) = \psi(\psi(0)\,2)$



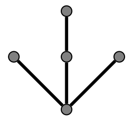
$\varepsilon_{\varepsilon_0{}^{\varepsilon_0}} = \phi_1(\phi_1(0)^{\phi_1(0)}) = \psi(\psi(0)^{\psi(0)})$



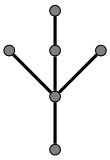
$\varepsilon_{\varepsilon_1} = \phi_1(\phi_1(1)) = \psi(\psi(1))$



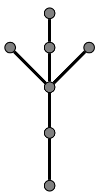
$\varepsilon_{\varepsilon_{\varepsilon_0}} = \phi_1(\phi_1(\phi_1(0))) = \psi(\psi(\psi(0)))$



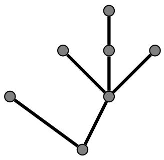
$\phi_2(0) = \psi(\Omega)$



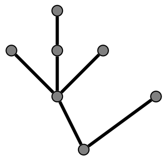
$\phi_2(0) + 1 = \psi(\Omega) + 1$



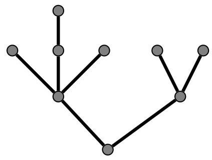
$\phi_2(0) + 2 = \psi(\Omega) + 2$



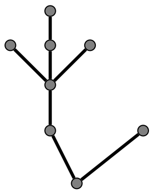
$\phi_2(0) + \omega = \psi(\Omega) + \omega$



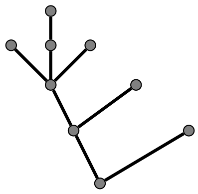
$\phi_2(0)\,2 = \psi(\Omega)\,2$



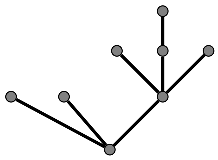
$\phi_2(0)\,\omega = \psi(\Omega)\,\omega$



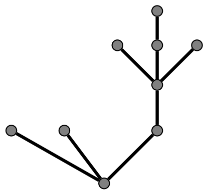
$\phi_2(0)^\omega = \psi(\Omega)^\omega$



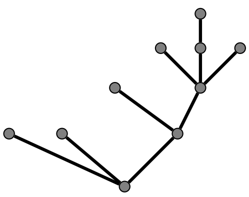
$\phi_2(0)^{\phi_2(0)} = \psi(\Omega)^{\psi(\Omega)}$



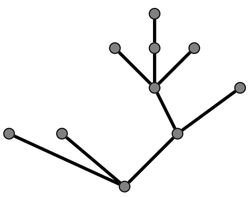
$\phi_1(\phi_2(0) + 1) = \psi(\Omega + 1)$



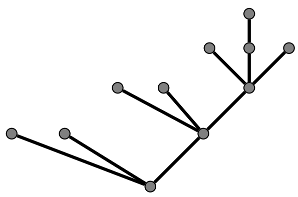
$\phi_1(\phi_2(0) + 2) = \psi(\Omega + 2)$



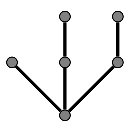
$\phi_1(\phi_2(0) + \omega) = \psi(\Omega + \omega)$



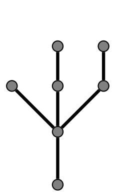
$\phi_1(\phi_2(0)\,2) = \psi(\Omega + \psi(\Omega))$



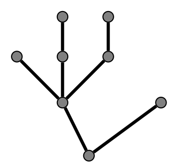
$\phi_1(\phi_1(\phi_2(0) + 1)) = \psi(\Omega + \psi(\Omega + 1))$



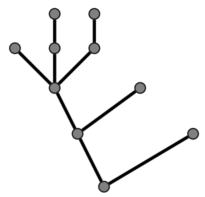
$\phi_2(1) = \psi(\Omega 2)$



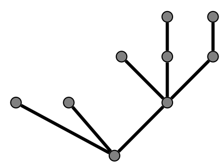
$\phi_2(1)+1=\psi(\Omega 2)+1$



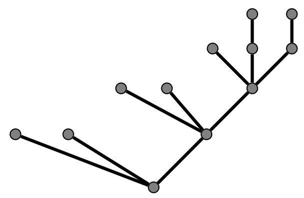
$\phi_2(1)\,2=\psi(\Omega 2)\,2$



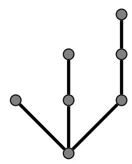
$\phi_2(1)^{\phi_2(1)}=\psi(\Omega 2)^{\psi(\Omega 2)}$



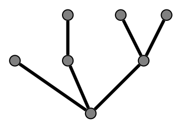
$\phi_1(\phi_2(1)+1)=\psi(\Omega 2+1)$



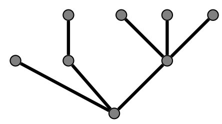
$\phi_1(\phi_1(\phi_2(1)+1))=\psi(\Omega 2+\psi(\Omega 2+1))$



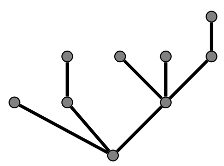
$\phi_2(2)=\psi(\Omega 3)$



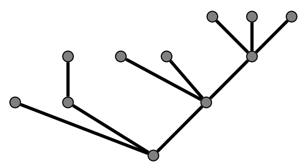
$\phi_2(\omega)=\psi(\Omega\omega)$



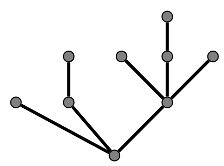
$\phi_2(\varepsilon_0)=\phi_2(\phi_1(0))=\psi(\Omega\psi(0))$



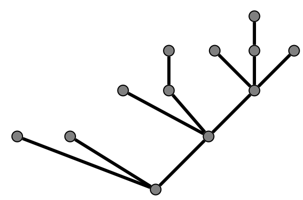
$\phi_2(\varepsilon_1)=\phi_2(\phi_1(1))=\psi(\Omega\psi(1))$



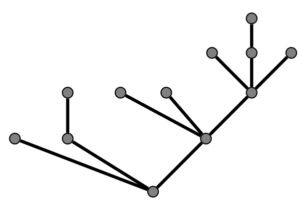
$\phi_2(\phi_1(\phi_1(0)))=\psi(\Omega\psi(\psi(0)))$



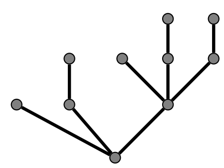
$\phi_2(\phi_2(0))=\psi(\Omega\psi(\Omega))$



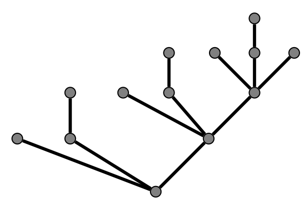
$\phi_1(\phi_2(\phi_2(0))+1)=\psi(\Omega\psi(\Omega)+1)$



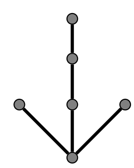
$\phi_2(\phi_1(\phi_2(0)+1))=\psi(\Omega\psi(\Omega+1))$



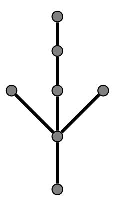
$\phi_2(\phi_2(1))=\psi(\Omega\psi(\Omega 2))$



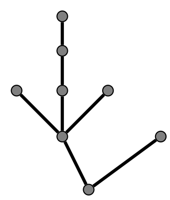
$\phi_2(\phi_2(\phi_2(0)))=\psi(\Omega\psi(\Omega\psi(\Omega)))$



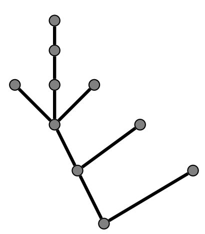
$\phi_3(0)=\psi(\Omega^2)$



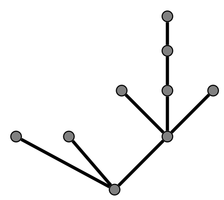
$\phi_3(0)+1=\psi(\Omega^2)+1$



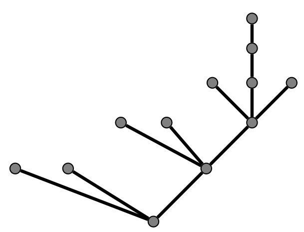
$\phi_3(0)\,2=\psi(\Omega^2)\,2$



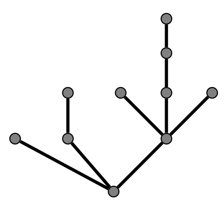
$\phi_3(0)^{\phi_3(0)}=\psi(\Omega^2)^{\psi(\Omega^2)}$



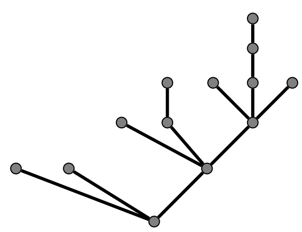
$\phi_1(\phi_3(0)+1)=\psi(\Omega^2+1)$



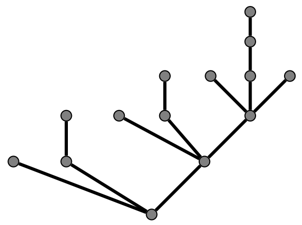
$\phi_1(\phi_1(\phi_3(0)+1))=\psi(\Omega^2+\psi(\Omega^2+1))$



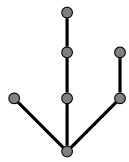
$\phi_2(\phi_3(0)+1)=\psi(\Omega^2+\Omega)$



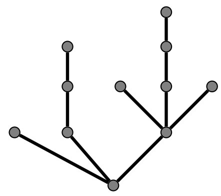
$\phi_1(\phi_2(\phi_3(0)+1)+1)=\psi(\Omega^2+\Omega+1)$



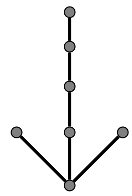
$\phi_2(\phi_2(\phi_3(0)+1)) = \psi(\Omega^2 + \Omega\psi(\Omega^2))$



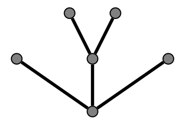
$\phi_3(1) = \psi(\Omega^2 2)$



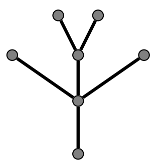
$\phi_3(\phi_3(0)) = \psi(\Omega^2\psi(\Omega^2))$



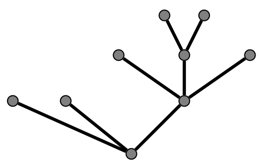
$\phi_4(0) = \psi(\Omega^3)$



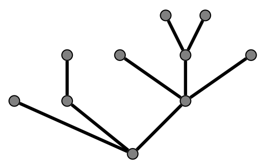
$\phi_\omega(0) = \psi(\Omega^\omega)$



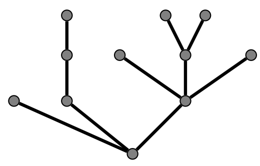
$\phi_\omega(0) + 1 = \psi(\Omega^\omega) + 1$



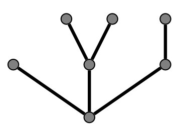
$\phi_1(\phi_\omega(0)+1) = \psi(\Omega^\omega + 1)$



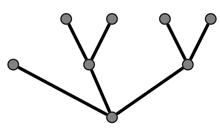
$\phi_2(\phi_\omega(0)+1) = \psi(\Omega^\omega + \Omega)$



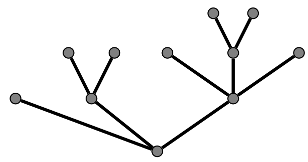
$\phi_3(\phi_\omega(0)+1) = \psi(\Omega^\omega + \Omega^2)$



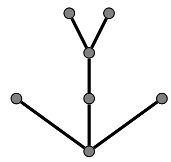
$\phi_\omega(1) = \psi(\Omega^\omega 2)$



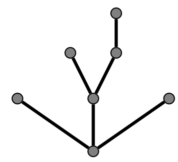
$\phi_\omega(\omega) = \psi(\Omega^\omega \omega)$



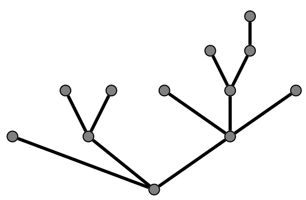
$\phi_\omega(\phi_\omega(0)) = \psi(\Omega^\omega\psi(\Omega^\omega))$



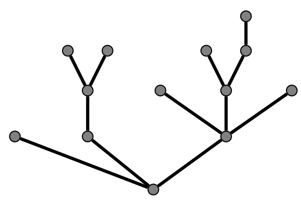
$\phi_{\omega+1}(0) = \psi(\Omega^{\omega+1})$



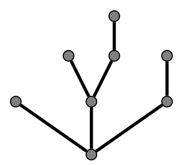
$\phi_{\omega 2}(0) = \psi(\Omega^{\omega 2})$



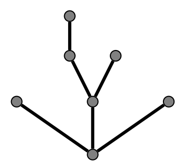
$\phi_\omega(\phi_{\omega 2}(0)+1) = \psi(\Omega^{\omega 2} + \Omega^\omega)$



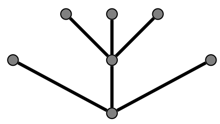
$\phi_{\omega+1}(\phi_{\omega 2}(0)+1) = \psi(\Omega^{\omega 2} + \Omega^{\omega+1})$



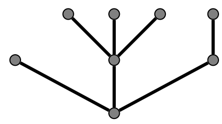
$\phi_{\omega 2}(1) = \psi(\Omega^{\omega 2}2)$



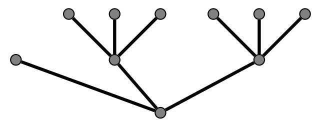
$\phi_{\omega^\omega}(0) = \psi(\Omega^{\omega^\omega})$



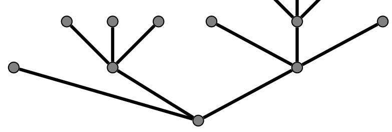
$\phi_{\varepsilon_0}(0) = \phi_{\phi_1(0)}(0) = \psi(\Omega^{\psi(0)})$



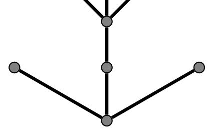
$\phi_{\varepsilon_0}(1) = \phi_{\phi_1(0)}(1) = \psi(\Omega^{\psi(0)}2)$



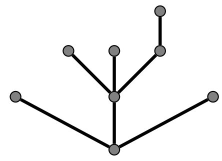
$\phi_{\varepsilon_0}(\varepsilon_0) = \phi_{\phi_1(0)}(\phi_1(0)) = \psi(\Omega^{\psi(0)}\psi(0))$



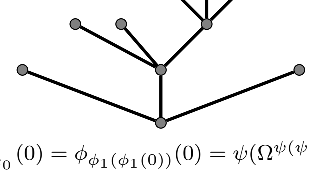
$\phi_{\phi_1(0)}(\phi_{\phi_1(0)}(0)) = \psi(\Omega^{\psi(0)}\psi(\Omega^{\psi(0)}))$



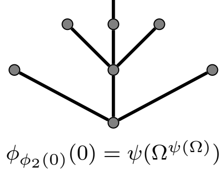
$\phi_{\varepsilon_0+1}(0) = \phi_{\phi_1(0)+1}(0) = \psi(\Omega^{\psi(0)+1})$



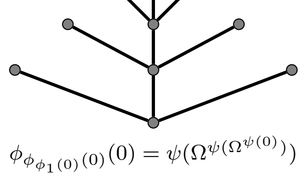
$\phi_{\varepsilon_1}(0) = \phi_{\phi_1(1)}(0) = \psi(\Omega^{\psi(1)})$



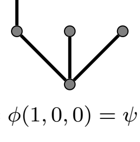
$\phi_{\varepsilon_{\varepsilon_0}}(0) = \phi_{\phi_1(\phi_1(0))}(0) = \psi(\Omega^{\psi(\psi(0))})$



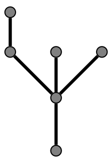
$\phi_{\phi_2(0)}(0) = \psi(\Omega^{\psi(\Omega)})$



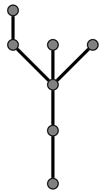
$\phi_{\phi_{\phi_1(0)}(0)}(0) = \psi(\Omega^{\psi(\Omega^{\psi(0)})})$



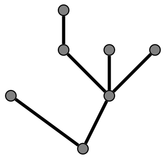
$\Gamma_0 = \phi(1,0,0) = \psi(\Omega^\Omega)$



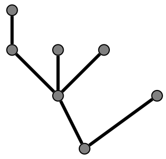
$\Gamma_0 + 1$



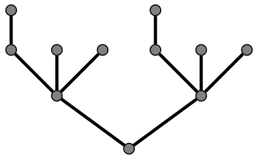
$\Gamma_0 + 2$



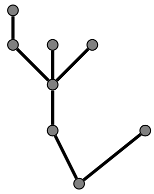
$\Gamma_0 + \omega$



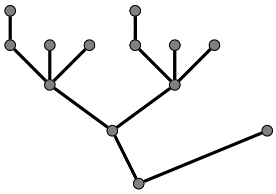
$\Gamma_0 2$



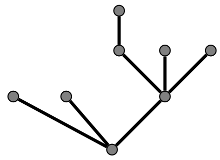
${\Gamma_0}^2$



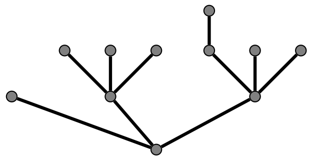
${\Gamma_0}^\omega$



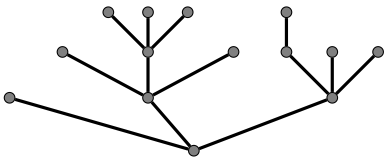
${\Gamma_0}^{\Gamma_0}$



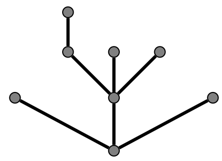
$\varepsilon_{\Gamma_0+1} = \phi(1, \phi(1,0,0)+1) = \psi(\Omega^\Omega + 1)$



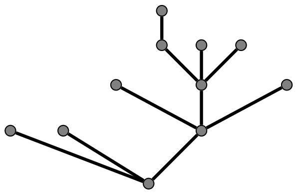
$\phi(\phi_1(0), \phi(1,0,0)+1) = \psi(\Omega^\Omega + \Omega^{\psi(0)})$



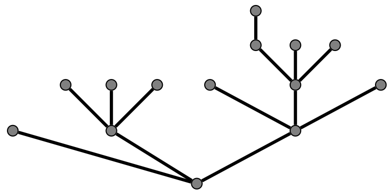
$\phi(\phi_{\phi_1(0)}(0), \phi(1,0,0)+1) = \psi(\Omega^\Omega + \Omega^{\psi(\Omega^{\psi(0)})})$



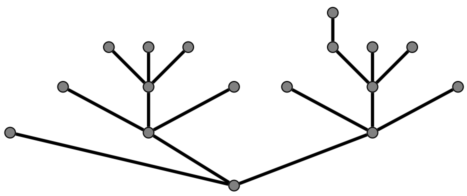
$\phi(\phi(1,0,0), 1) = \psi(\Omega^\Omega + \Omega^{\psi(\Omega^\Omega)})$



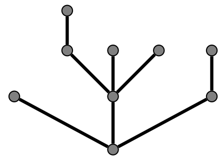
$\phi(1, \phi(\phi(1,0,0),1)+1) = \psi(\Omega^\Omega + \Omega^{\psi(\Omega^\Omega)} + 1)$



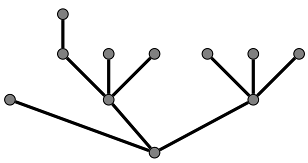
$\phi(\phi_1(0), \phi(\phi(1,0,0),1)+1) = \psi(\Omega^\Omega + \Omega^{\psi(\Omega^\Omega)} + \Omega^{\psi(0)})$



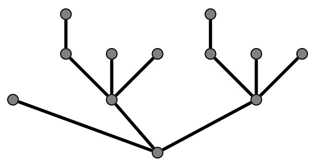
$\phi(\phi_{\phi_1(0)}(0), \phi(\phi(1,0,0),1)+1) = \psi(\Omega^\Omega + \Omega^{\psi(\Omega^\Omega)} + \Omega^{\psi(\Omega^{\psi(0)})})$



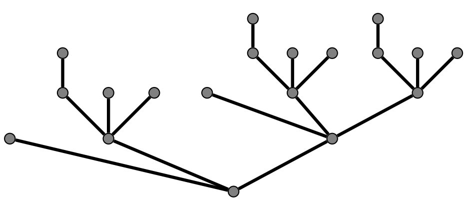
$\phi(\phi(1,0,0), 2) = \psi(\Omega^\Omega + \Omega^{\psi(\Omega^\Omega)} 2)$



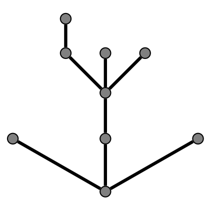
$\phi(\phi(1,0,0), \phi_1(0)) = \psi(\Omega^\Omega + \Omega^{\psi(\Omega^\Omega)} \psi(0))$



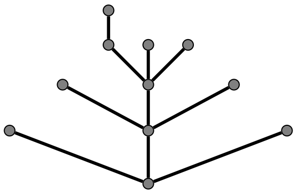
$\phi(\phi(1,0,0), \phi(1,0,0)) = \psi(\Omega^\Omega + \Omega^{\psi(\Omega^\Omega)} \psi(\Omega^\Omega))$



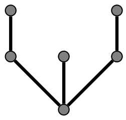
$\phi(\phi(1,0,0), \phi(\phi(1,0,0), \phi(1,0,0))) = \psi(\Omega^\Omega + \Omega^{\psi(\Omega^\Omega)} \psi(\Omega^\Omega + \Omega^{\psi(\Omega^\Omega)} \psi(\Omega^\Omega)))$



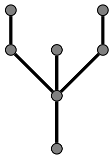
$\phi(\phi(1,0,0)+1, 0) = \psi(\Omega^\Omega + \Omega^{\psi(\Omega^\Omega)+1})$



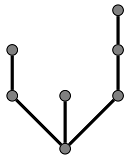
$\phi(\phi(\phi(1,0,0),1),0) = \psi(\Omega^{\Omega} + \Omega^{\psi(\Omega^{\Omega} + \Omega^{\psi(\Omega^{\Omega})})})$



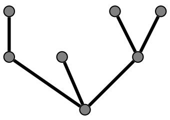
$\Gamma_1 = \phi(1,0,1) = \psi(\Omega^{\Omega} 2)$



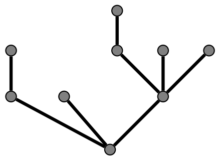
$\Gamma_1 + 1$



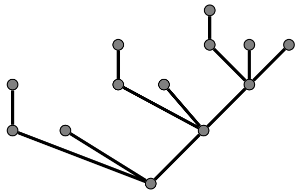
$\Gamma_2 = \phi(1,0,2) = \psi(\Omega^{\Omega} 3)$



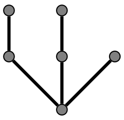
$\Gamma_\omega = \phi(1,0,\omega) = \psi(\Omega^{\Omega}\omega)$



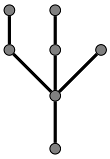
$\Gamma_{\Gamma_0} = \phi(1,0,\phi(1,0,0)) = \psi(\Omega^{\Omega}\psi(\Omega^{\Omega}))$



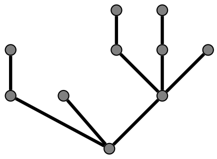
$\Gamma_{\Gamma_{\Gamma_0}} = \phi(1,0,\phi(1,0,\phi(1,0,0))) = \psi(\Omega^{\Omega}\psi(\Omega^{\Omega}\psi(\Omega^{\Omega})))$



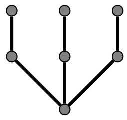
$\phi(1,1,0) = \psi(\Omega^{\Omega+1})$



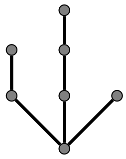
$\phi(1,1,0) + 1$



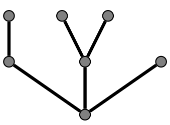
$\phi(1,0,\phi(1,1,0)+1) = \psi(\Omega^{\Omega+1} + \Omega^{\Omega})$



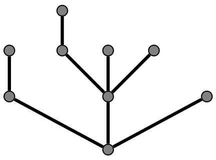
$\phi(1,1,1) = \psi(\Omega^{\Omega+1}2)$



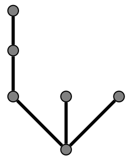
$\phi(1,2,0) = \psi(\Omega^{\Omega+2})$



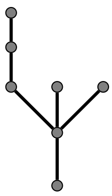
$\phi(1,\omega,0) = \psi(\Omega^{\Omega+\omega})$



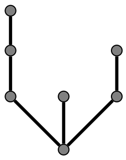
$\phi(1,\phi(1,0,0),0) = \psi(\Omega^{\Omega+\psi(\Omega^{\Omega})})$



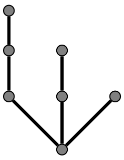
$\phi(2,0,0) = \psi(\Omega^{\Omega 2})$



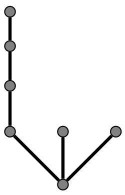
$\phi(2,0,0) + 1$



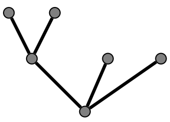
$\phi(2,0,1) = \psi(\Omega^{\Omega 2}2)$



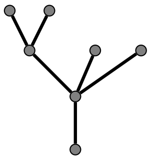
$\phi(2,1,0) = \psi(\Omega^{\Omega 2+1})$



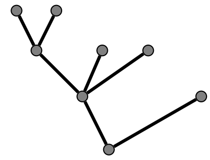
$\phi(3,0,0) = \psi(\Omega^{\Omega 3})$



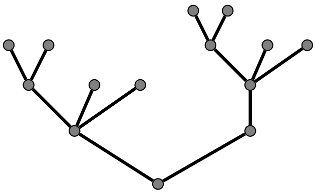
$\phi(\omega,0,0) = \psi(\Omega^{\Omega\omega})$



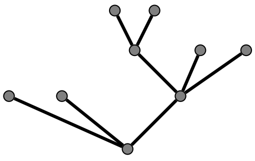
$\phi(\omega,0,0) + 1$



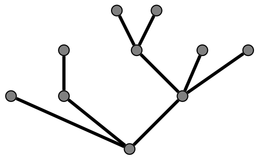
$\phi(\omega,0,0)\,2$



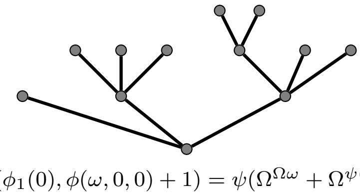
$\phi(\omega,0,0)^{\phi(\omega,0,0)}$



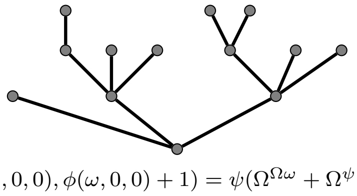
$\phi(1,\phi(\omega,0,0)+1) = \psi(\Omega^{\Omega\omega} + 1)$



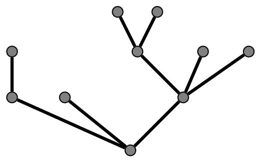
$\phi(2,\phi(\omega,0,0)+1) = \psi(\Omega^{\Omega\omega} + \Omega)$



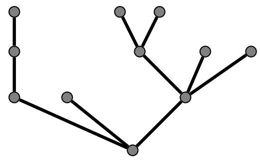
$\phi(\phi_1(0),\phi(\omega,0,0)+1) = \psi(\Omega^{\Omega\omega} + \Omega^{\psi(0)})$



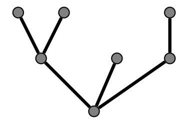
$\phi(\phi(1,0,0),\phi(\omega,0,0)+1) = \psi(\Omega^{\Omega\omega} + \Omega^{\psi(\Omega^{\Omega})})$



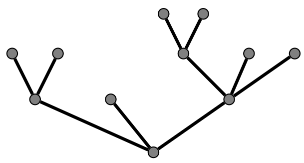
$\phi(1,0,\phi(\omega,0,0)+1)=\psi(\Omega^{\Omega\omega}+\Omega^{\Omega})$



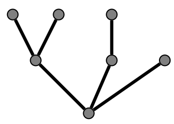
$\phi(2,0,\phi(\omega,0,0)+1)=\psi(\Omega^{\Omega\omega}+\Omega^{\Omega 2})$



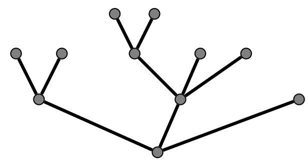
$\phi(\omega,0,1)=\psi(\Omega^{\Omega\omega}2)$



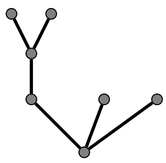
$\phi(\omega,0,\phi(\omega,0,0))=\psi(\Omega^{\Omega\omega}\psi(\Omega^{\Omega\omega}))$



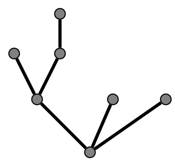
$\phi(\omega,1,0)=\psi(\Omega^{\Omega\omega+1})$



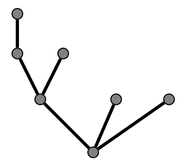
$\phi(\omega,\phi(\omega,0,0),0)=\psi(\Omega^{\Omega\omega+\psi(\Omega^{\Omega\omega})})$



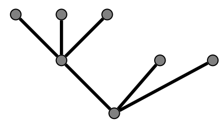
$\phi(\omega+1,0,0)=\psi(\Omega^{\Omega(\omega+1)})$



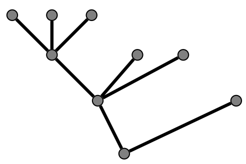
$\phi(\omega 2,0,0)=\psi(\Omega^{\Omega\omega 2})$



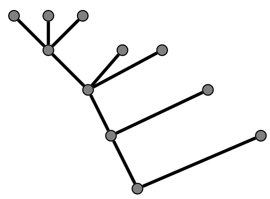
$\phi(\omega^{\omega},0,0)=\psi(\Omega^{\Omega\omega^{\omega}})$



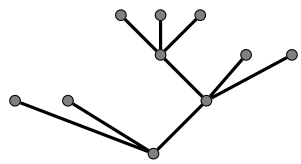
$\phi(\phi_1(0),0,0)=\psi(\Omega^{\Omega\psi(0)})$



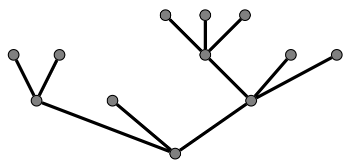
$\phi(\phi_1(0),0,0)\,2$



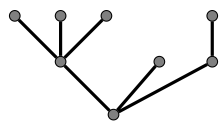
$\phi(\phi_1(0),0,0)^{\phi(\phi_1(0),0,0)}$



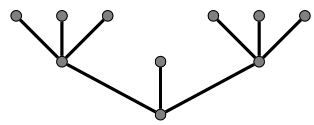
$\phi(1,\phi(\phi_1(0),0,0)+1)=\psi(\Omega^{\Omega\psi(0)}+1)$



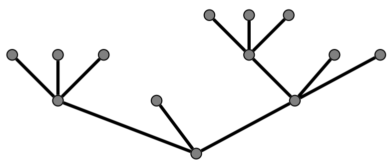
$\phi(\omega,\phi(\phi_1(0),0,0)+1)=\psi(\Omega^{\Omega\psi(0)}+\Omega^{\omega})$



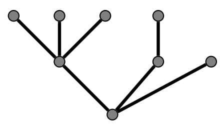
$\phi(\phi_1(0),0,1)=\psi(\Omega^{\Omega\psi(0)}2)$



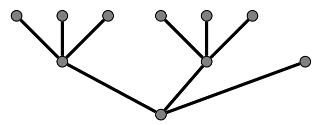
$\phi(\phi_1(0),0,\phi_1(0))=\psi(\Omega^{\Omega\psi(0)}\psi(0))$



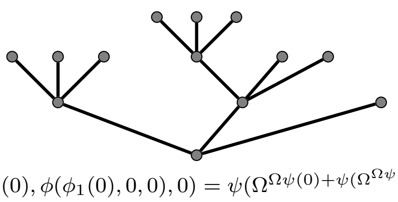
$\phi(\phi_1(0),0,\phi(\phi_1(0),0,0))=\psi(\Omega^{\Omega\psi(0)}\psi(\Omega^{\Omega\psi(0)}))$



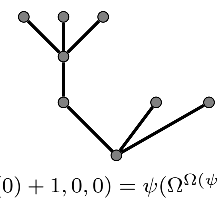
$\phi(\phi_1(0),1,0)=\psi(\Omega^{\Omega\psi(0)+1})$



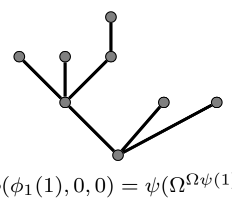
$\phi(\phi_1(0),\phi_1(0),0)=\psi(\Omega^{\Omega\psi(0)+\psi(0)})$



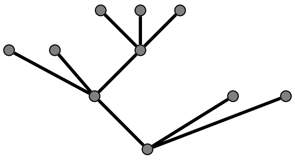
$\phi(\phi_1(0),\phi(\phi_1(0),0,0),0)=\psi(\Omega^{\Omega\psi(0)+\psi(\Omega^{\Omega\psi(0)})})$



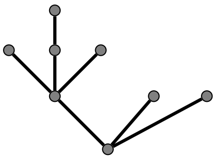
$\phi(\phi_1(0)+1,0,0)=\psi(\Omega^{\Omega(\psi(0)+1)})$



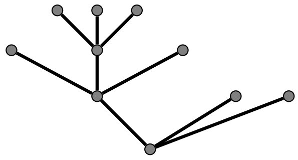
$\phi(\phi_1(1),0,0)=\psi(\Omega^{\Omega\psi(1)})$



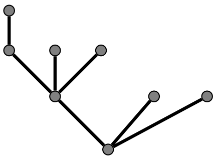
$\phi(\phi_1(\phi_1(0)),0,0)=\psi(\Omega^{\Omega\psi(\psi(0))})$



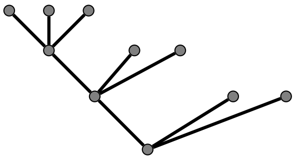
$\phi(\phi_2(0),0,0)=\psi(\Omega^{\Omega\psi(\Omega)})$



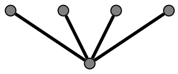
$\phi(\phi_{\phi_1(0)}(0),0,0)=\psi(\Omega^{\Omega\psi(\Omega^{\psi(0)})})$



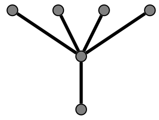
$\phi(\phi(1,0,0),0,0)=\psi(\Omega^{\Omega\psi(\Omega^{\Omega})})$



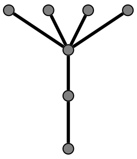
$\phi(\phi(\phi_1(0),0,0),0,0)=\psi(\Omega^{\Omega\psi(\Omega^{\Omega\psi(0)})})$



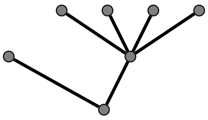
$\phi(1,0,0,0)=\psi(\Omega^{\Omega^2})$



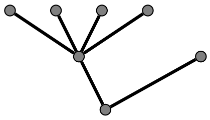
$\phi(1,0,0,0)+1$



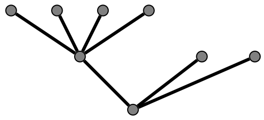
$\phi(1,0,0,0)+2$



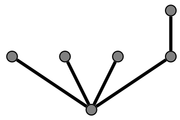
$\phi(1,0,0,0)+\omega$



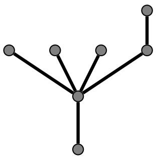
$\phi(1,0,0,0)2$



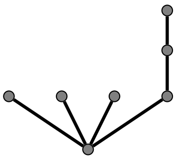
$\phi(1,0,\phi(1,0,0,0)+1)=\psi(\Omega^{\Omega^2}+\Omega^{\Omega})$



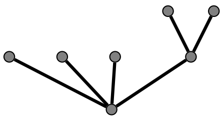
$\phi(1,0,0,1)=\psi(\Omega^{\Omega^2}2)$



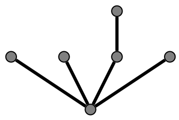
$\phi(1,0,0,1)+1$



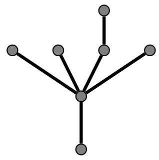
$\phi(1,0,0,2)=\psi(\Omega^{\Omega^2}3)$



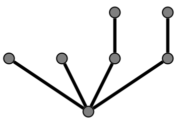
$\phi(1,0,0,\omega)=\psi(\Omega^{\Omega^2}\omega)$



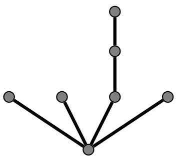
$\phi(1,0,1,0)=\psi(\Omega^{\Omega^2+1})$



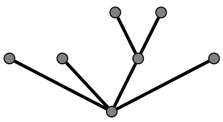
$\phi(1,0,1,0)+1$



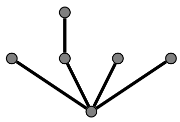
$\phi(1,0,1,1)=\psi(\Omega^{\Omega^2+1}2)$



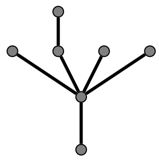
$\phi(1,0,2,0)=\psi(\Omega^{\Omega^2+2})$



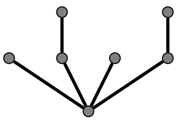
$\phi(1,0,\omega,0)=\psi(\Omega^{\Omega^2+\omega})$



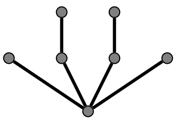
$\phi(1,1,0,0)=\psi(\Omega^{\Omega^2+\Omega})$



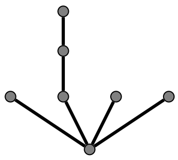
$\phi(1,1,0,0)+1$



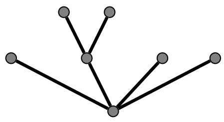
$\phi(1,1,0,1)=\psi(\Omega^{\Omega^2+\Omega}2)$



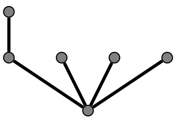
$\phi(1,1,1,0)=\psi(\Omega^{\Omega^2+\Omega+1})$



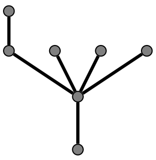
$\phi(1,2,0,0)=\psi(\Omega^{\Omega^2+\Omega 2})$



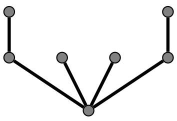
$\phi(1,\omega,0,0)=\psi(\Omega^{\Omega^2+\Omega\omega})$



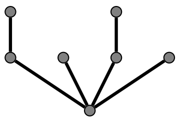
$\phi(2,0,0,0)=\psi(\Omega^{\Omega^2 2})$



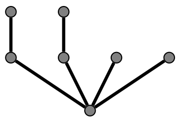
$\phi(2,0,0,0)+1$



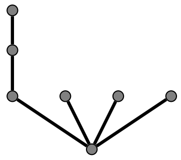
$\phi(2,0,0,1)=\psi(\Omega^{\Omega^2 2}2)$



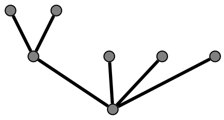
$\phi(2,0,1,0)=\psi(\Omega^{\Omega^2 2+1})$



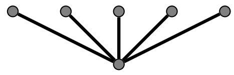
$\phi(2,1,0,0)=\psi(\Omega^{\Omega^2 2+\Omega})$



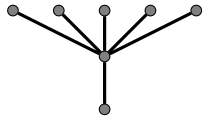
$\phi(3,0,0,0) = \psi(\Omega^{\Omega^2 3})$



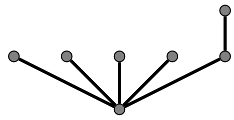
$\phi(\omega,0,0,0) = \psi(\Omega^{\Omega^2 \omega})$



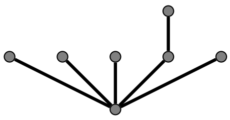
$\phi(1,0,0,0,0) = \psi(\Omega^{\Omega^3})$



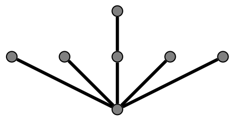
$\phi(1,0,0,0,0) + 1$



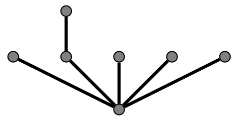
$\phi(1,0,0,0,1) = \psi(\Omega^{\Omega^3} 2)$



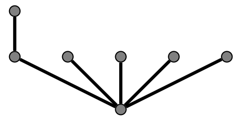
$\phi(1,0,0,1,0) = \psi(\Omega^{\Omega^3+1})$



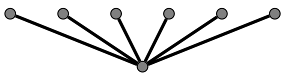
$\phi(1,0,1,0,0) = \psi(\Omega^{\Omega^3+\Omega})$



$\phi(1,1,0,0,0) = \psi(\Omega^{\Omega^3+\Omega^2})$



$\phi(2,0,0,0,0) = \psi(\Omega^{\Omega^3 2})$



$\phi(1,0,0,0,0,0) = \psi(\Omega^{\Omega^4})$

Conventions and notations:

- The order on finite rooted trees is recursively defined as follows: a tree A is less than a tree B, written $A \prec B$, iff:
  - *either* there is some child (=immediate subtree) B' of B such that $A \preceq B'$,
  - *or* the following two conditions hold: every child A' of A satisfies $A' \prec B$, *and* the list of children of A is lexicographically less than the list of children of B for the order $\prec$ (with the leftmost children having the most weight, i.e., either B has more children than A, or if A' and B' are the leftmost children on which they differ then $A' \prec B'$).

  This is a well-order.
- ω is the smallest infinite ordinal.
- Ordinal addition, multiplication and exponentiation are defined as usual:
  - (i) $\alpha + 0 = \alpha$
  - (ii) $\alpha + 1$ is the successor of α.
  - (iii) $\alpha + (\beta + 1) = (\alpha + \beta) + 1$
  - (iv) If δ is limit then $\alpha + \delta$ is the limit of the $\alpha + \beta$ for $\beta < \delta$.
  - (v) $\alpha \cdot 0 = 0$
  - (vi) $\alpha(\beta + 1) = \alpha\beta + \alpha$
  - (vii) $\alpha^0 = 1$
  - (viii) $\alpha^{\beta+1} = \alpha^\beta \alpha$
- $\phi_1(\alpha) = \varepsilon_\alpha$ is the α-th fixed point of $\xi \mapsto \omega^\xi$ (with ε_0 the smallest: it is the limit of ω, ω^ω, ω^{ω^ω}, etc.).
- *The Veblen hierarchy:* For any $\beta > 1$, let $\phi_\beta(\alpha)$ be the α-th common fixed point of ϕ_γ for all $0 < \gamma < \beta$ (with $\phi_\beta(0)$ the smallest). Note:
  - (i) $\phi_\beta(\alpha)$ is continuous in α for all β but it is *not* continuous in β except for $\alpha = 0$.
  - (ii) $\phi_{\beta+1}(0)$ is the limit of $\phi_\beta(0)$, $\phi_\beta(\phi_\beta(0))$, $\phi_\beta(\phi_\beta(\phi_\beta(0)))$, etc.
  - (iii) $\phi_{\beta+1}(\alpha + 1)$ is the limit of $\phi_{\beta+1}(\alpha) + 1$, $\phi_\beta(\phi_{\beta+1}(\alpha) + 1)$, $\phi_\beta(\phi_\beta(\phi_{\beta+1}(\alpha) + 1))$, etc.
  - (iv) If δ is limit (and β is arbitrary) then $\phi_\beta(\delta)$ is the limit of the $\phi_\beta(\xi)$ for $\xi < \delta$.
  - (v) If δ is limit then $\phi_\delta(0)$ is the limit of the $\phi_\gamma(0)$ for $\gamma < \delta$.
  - (vi) If δ is limit then $\phi_\delta(\alpha + 1)$ is the limit of the $\phi_\gamma(\phi_\delta(\alpha) + 1)$ for $\gamma < \delta$.

- $\phi(1,0,\alpha) = \Gamma_\alpha$ is the α-th fixed point of $\xi \mapsto \phi_\xi(0)$ (with Γ_0 the smallest: it is the limit of $\phi_1(0)$, $\phi_{\phi_1(0)}(0)$ $\phi_{\phi_{\phi_1(0)}(0)}(0)$, etc.; this is known as the Feferman-Schütte ordinal).
- More generally, let $\phi(\beta,\alpha) = \phi_\beta(\alpha)$ (we only use the notation with a subscript for values less than Γ_0) and define ϕ of finitely many variables by: $\phi(\beta_n, \beta_{n-1}, \ldots, \beta_r, 0, 0, \ldots, 0, \alpha)$ is the α-th fixed point of all the $\xi \mapsto \phi(\gamma_n, \gamma_{n-1}, \ldots, \gamma_r, \xi, 0, 0, \ldots, 0)$ for $(\gamma_n, \ldots, \gamma_r)$ lexicographically less than $(\beta_n, \ldots, \beta_r)$ (the leftmost variable having the most weight), with the convention that $\phi(0, \beta_{n-1}, \ldots, \beta_0) = \phi(\beta_{n-1}, \ldots, \beta_0)$.
- Ω is the first uncountable ordinal.
- *A collapsing function:* $\psi(\alpha)$ is defined inductively as the smallest ordinal not expressible from 0, 1, ω and Ω using addition, multiplication, exponentiation, and application of the function ψ itself to ordinals less than α. Or, more rigorously:
  - Assume ψ has been defined for all ordinals $\beta < \alpha$.
  - Let $C(\alpha)$ be the set of ordinals constructed starting from 0, 1, ω and Ω by recursively applying the following functions: ordinal addition, multiplication and exponentiation and the function $\psi\restriction_\alpha$, i.e., the restriction of ψ to ordinals $\beta < \alpha$. Formally, we define $C(\alpha)_0 = \{0, 1, \omega, \Omega\}$ and inductively $C(\alpha)_{n+1} = C(\alpha)_n \cup \{\beta_1 + \beta_2,\ \beta_1\beta_2,\ {\beta_1}^{\beta_2} :\ \beta_1, \beta_2 \in C(\alpha)_n\} \cup \{\psi(\beta) : \beta \in C(\alpha)_n \wedge \beta < \alpha\}$ for all natural numbers n, and we let $C(\alpha)$ be the union of the $C(\alpha)_n$ for all n.
  - Define $\psi(\alpha)$ as the smallest ordinal not in $C(\alpha)$.

  It turns out that:

  (i) ψ is continuous and non-decreasing.

  (ii) The range of ψ is precisely the set of ε-numbers (i.e., fixed points of $\xi \mapsto \omega^\xi$) up to the Bachmann-Howard ordinal.

  (iii) $C(\alpha)$ consists exactly of those ordinals whose iterated base Ω representation only has pieces less than $\psi(\alpha)$; in particular, $C(\alpha) \cap \Omega = \alpha$.

  (iv) $\psi(\alpha+1)$ is always equal either to the first ε-number after $\psi(\alpha)$—this happens when $\alpha \in C(\alpha)$—or else to $\psi(\alpha)$—which happens when $\alpha \notin C(\alpha)$.

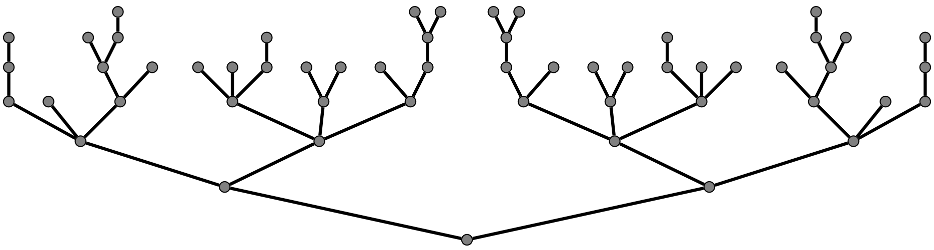
  (v) If δ is limit then $\psi(\delta)$ is the limit of the $\psi(\xi)$ for $\xi < \delta$.

  (vi) $\psi(\alpha + \Omega)$ is the first fixed point of $\xi \mapsto \psi(\alpha + \xi)$.

  (vii) $\psi(\alpha\Omega)$ is the first fixed point of $\xi \mapsto \psi(\alpha\xi)$.

  (viii) $\psi(\alpha^\Omega)$ is the first fixed point of $\xi \mapsto \psi(\alpha^\xi)$.

  (ix) $\psi(\alpha^{\beta^\Omega})$ is the first fixed point of $\xi \mapsto \psi(\alpha^{\beta^\xi})$.
- Certain ordinals have special names:
  - $\psi(\Omega^\Omega) = \phi(1,0,0) = \Gamma_0$ is the "Feferman-Schütte ordinal": it is the set of ordinals constructed starting from 0, 1, ω by recursively applying ordinal addition, multiplication and exponentiation and the Veblen functions $\phi_\beta(\alpha)$ of two variables;
  - $\psi(\Omega^{\Omega^\omega})$ is the "small" Veblen ordinal: it is the set of ordinals constructed starting from 0, 1, ω by recursively applying ordinal addition, multiplication and exponentiation and the Veblen functions $\phi(\cdots)$ of finitely many variables, and it is also the length of the ordering $\prec$ of finite rooted trees;
  - $\psi(\Omega^{\Omega^\Omega})$ is the "large" Veblen ordinal (it can be defined similarly to the "small" Veblen ordinal using a generalization of $\phi(\cdots)$ to transfinitely many variables all but finitely many of which are zero);
  - $\psi(\varepsilon_{\Omega+1})$ is the Bachmann-Howard ordinal.



α_{example}

$$\begin{aligned}\alpha_{\text{example}} &= \phi(\phi_1(1), \omega, \omega^2 + \omega)^{\phi(2,0,\omega^{\omega^{\omega 2}})}\, \phi(\omega^{\omega^{\omega+1}}, \omega, \phi(1,0,0))\, \phi(\omega^{\omega} + \omega, 0, 2) \\ &= \psi(\Omega^{\Omega\psi(1)+\omega}(\omega^2 + \omega))^{\psi(\Omega^{\Omega 2}\omega^{\omega^{\omega 2}})}\, \psi(\Omega^{\Omega\omega^{\omega^{\omega+1}}+\omega}\psi(\Omega^{\Omega}))\, \psi(\Omega^{\Omega(\omega^{\omega}+\omega)}3)\end{aligned}$$