On the decomposition of metric spaces in disjoint sets

Wacław Sierpiński (Warszawa)

The goal of this note is to prove the following theorem for arbitrary metric spaces (separable or not), and to draw a few consequences of it:

Theorem. *Every metric space M in which every non-empty open set contains $\geq \mathfrak{m} \geq \aleph_0$ points, is sum of $\mathfrak{m}$ disjoint sets, each of which contains $\geq \mathfrak{m}$ points of every non-empty open set $\subset M$.*

Proof. A point p of a set E contained in a metric space M is called *point of order* $\mathfrak{n}$ of E when every open sphere [i.e., "ball" — translator's note] with centre p and sufficiently small radius contains precisely $\mathfrak{n}$ points of E. (It is easy to prove that every point p of a set E is of some precise order $\mathfrak{n}$: namely $\mathfrak{n}$ is the smallest of all the cardinal numbers $\overline{\overline{EU}}$, where U is any open sphere of centre p.)

A set $E \subset M$ is called *homogeneous* of order $\mathfrak{n}$ when every point of E is a point of order $\mathfrak{n}$ of E[1].

Lemma 1. *If M is a metric space which is dense-in-itself, and U a non-empty open subset $\subset M$, there exists an open sphere S, $\varnothing \neq S \subset U$, which is a homogeneous set.*

Proof. Let $\mathfrak{n}$ be the smallest of all the cardinal numbers $\overline{\overline{S}}$, where S is a non-empty open sphere $\subset U$. So there exists an open sphere $\subset U$ such that $\overline{\overline{S}} = \mathfrak{n}$. If the set S were not homogeneous, there would exist a non-empty open sphere $S_1 \subset S$ such that $\overline{\overline{S_1}} \neq \mathfrak{n}$, so $\overline{\overline{S_1}} < \mathfrak{n}$, contrary to the definition of the number $\mathfrak{n}$. The lemma is thus proved.

Lemma 2. *Every metric space M with cardinality $\mathfrak{n} \geq \aleph_0$ in which every point is of order $\mathfrak{n}$, is sum of $\mathfrak{n}$ disjoint sets such that every neighbourhood of every point of M contains $\mathfrak{n}$ points of each of the sets.*

Proof. Let M be a metric space of cardinality $\mathfrak{n} = \aleph_\alpha$ in which every point is of order $\mathfrak{n}$. Let F be the family of all the open spheres whose centre is a point of M and radius $= \frac{1}{n}$, where $n = 1, 2, \ldots$. The family F is obviously of cardinality $\mathfrak{n} \cdot \aleph_0 = \aleph_\alpha \aleph_0 = \aleph_\alpha$ and, every point of M being of order $\mathfrak{n}$, every sphere of the family F is a set of cardinality $\mathfrak{n} = \aleph_\alpha$. But, as Mr. Kuratowski[2] proved, if $F = \{E_\xi\}_{\xi<\omega_\alpha}$ is a transfinite sequence of type ω_α of sets of cardinality $\aleph_\alpha$, there exists a transfinite sequence of type ω_α of disjoint sets of cardinality $\aleph_\alpha$, $\{H_\xi\}_{\xi<\omega_\alpha}$, such that $H_\xi \subset E_\xi$ for $\xi < \omega_\alpha$. But H_ξ, being a set of cardinality $\aleph_\alpha$, is (because ${\aleph_\alpha}^2 = \aleph_\alpha$) sum of $\aleph_\alpha$ disjoint sets with cardinality $\aleph_\alpha$; say $H_\xi = \sum_{\eta<\omega_\alpha} H_{\xi,\eta}$. Let $P_\eta = \sum_{\xi<\omega_\alpha} H_{\xi,\eta}$ for $\eta < \omega_\alpha$. Since $H_{\xi,\eta} \subset H_\xi \subset E_\xi$ for $\xi < \omega_\alpha$, $\eta < \omega_\alpha$, the set P_η contains $\aleph_\alpha$ points of every sphere with centre p and radius $1/n$ for $p \in M$, $n = 1, 2, \ldots$. Every neighbourhood of every point of M therefore contains $\mathfrak{n}$ points of each of the sets P_η for $\eta < \omega_\alpha$. But the sets P_η $(\eta < \omega_\alpha)$ are obviously disjoint. Lemma 2 is thus proved.

Let M be a given metric space in which every open set contains $\geq \mathfrak{m}$ points; let

$$x_1, x_2, \ldots, x_\omega, x_{\omega+1}, \ldots, x_\xi, \ldots \tag{1}$$

be a transfinite sequence consisting of all the points of M, and

$$S_1, S_2, \ldots, S_\omega, S_{\omega+1}, \ldots, S_\xi, \ldots \tag{2}$$

[1] This notion is due to G. Cantor: Acta Math. **7** (1885), p. 118; cf. also W. Sierpiński, Fund. Math. **1**, p. 28.

[2] See Fund. Math. **34** (1947), p. 35, lemma 1.

be a transfinite sequence consisting of all the non-empty open spheres whose centres are the points of M. We will define by transfinite induction a transfinite sequence of spheres

$$T_1, T_2, \dots, T_\omega, T_{\omega+1}, \dots, T_\xi, \dots \tag{3}$$

as follows.

The space M is evidently dense-in-itself; from lemma 1, there exists a non-empty open sphere $S \subset M$ such that the set S is homogeneous, say of order $\mathfrak{n}$. If p is a point of S, there exists a sphere $S^* \subset S$ of centre p, and such that S^* is homogeneous of order $\mathfrak{n}$ and cardinality $\mathfrak{n}$. So in the sequence (2) there exists a first term S_λ such that S_λ is a homogeneous set of order $\overline{\overline{S_\lambda}}$. Let $T_1 = S_\lambda$.

Let α be an ordinal number < 1 [sic] and suppose we have already defined all the spheres T_ξ where $\xi < \alpha$. Let $H_\alpha = \overline{\sum_{\xi<\alpha} T_\xi}$ (where $\overline{E}$ denotes the closure of the set E). If $M \subset H_\alpha$, the definition of the sequence (3) is finished (it is then of type α). If $M - H_\alpha \neq \varnothing$, the set $M - H_\alpha$ is open and non-empty and, as above, because of lemma 1, we can conclude that there exists in sequence (2) a first term S_μ [contained in $M - H_\alpha$] such that S_μ is a homogeneous set of cardinality [order] $\overline{\overline{S_\mu}}$. We let $T_\alpha = S_\mu$.

The sequence (3) is thus defined by transfinite induction. Let ϑ be its type; the spheres T_ξ ($\xi < \vartheta$) are evidently disjoint and we have $M \subset \overline{\bigcup_{\xi<\vartheta} T_\xi}$. The set T_ξ is homogeneous of order $\overline{\overline{T_\xi}} \geq \mathfrak{m} = \aleph_\gamma$. According to lemma 2 we thus have $T_\xi = \sum_{\eta<\omega_\gamma} K_{\xi,\eta}$, where $K_{\gamma,\xi}$ ($\eta < \omega_\gamma$) are disjoint sets each of which contains $\overline{\overline{T_\xi}} \geq \mathfrak{m}$ points of each neighbourhood of any point of T_ξ. Let $K_\eta = \sum_{\xi<\vartheta} K_{\xi,\eta}$ for $\eta < \omega_\gamma$. I claim that the sets K_η ($\eta < \omega_\gamma$) satisfy the theorem.

The sets $K_{\xi,\eta}$ ($\xi < \vartheta$, $\eta < \omega_\gamma$) being disjoint, the sets K_η ($\eta < \omega_\gamma$) are also. Let U be a non-empty open set $\subset M$. As $U \subset M \subset \overline{\sum_{\xi<\vartheta} T_\xi}$ and U is open and non-empty, there exists a point p of the set $\sum_{\xi<\vartheta} T_\xi$ such that $p \in U$, and there exists an ordinal $\nu < \vartheta$ such that $p \in T_\nu$. Now, the set $K_{\nu,\eta}$ contains $\overline{\overline{T_\nu}} \geq \mathfrak{m}$ points of each neighbourhood of any point of T_ν, so $\mathfrak{m}$ points of U. The set $K_\eta \subset K_{\nu,\eta}$ thus contains $\mathfrak{m}$ points of U. So the theorem is proved.

For $\mathfrak{m} = \aleph_0$, our theorem immediately gives the following

Corollary 1. *Every dense-in-itself metric space M is sum of an infinite sequence of disjoint dense subsets of M.*

I have given elsewhere[3] a direct proof of this corollary.

Since condensed sets coincide with sets in which every point is of order $\geq \aleph_1$, it follows immediately from our theorem (for $\mathfrak{m} = \aleph_1$) that:

Corollary 2. *Every condensed metric space M is sum of an uncountable infinity of condensed metric sets which are dense in M.*

[3] Proceedings of the Benares Mathematical Society New Series Vol. VII (1945), pp. 29–31; cf. E. Hewitt, Duke Math. Journ. **10** (1943), pp. 309–333.