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Galois Cohomology

translated from the French by Patrick Ion

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Foreword

This volume is an English translation of "Cohomologie Galoisienne". The original edition (Springer LN5, 1964) was based on the notes, written with the help of Michel Raynaud, of a course I gave at the Collège de France in 1962–1963. In the present edition there are numerous additions and one suppression: Verdier's text on the duality of profinite groups. The most important addition is the photographic reproduction of R. Steinberg's "Regular elements of semisimple algebraic groups", Publ. Math. I.H.E.S., 1965. I am very grateful to him, and to I.H.E.S., for having authorized this reproduction.

Other additions include:

- A proof of the Golod-Shafarevich inequality (Chap. I, App. 2).
- The "résumé de cours" of my 1991–1992 lectures at the Collège de France on Galois cohomology of $k(T)$ (Chap. II, App.).
- The "résumé de cours" of my 1990–1991 lectures at the Collège de France on Galois cohomology of semisimple groups, and its relation with abelian cohomology, especially in dimension 3 (Chap. III, App. 2).

The bibliography has been extended, open questions have been updated (as far as possible) and several exercises have been added.

In order to facilitate references, the numbering of propositions, lemmas and theorems has been kept as in the original 1964 text.

Jean-Pierre Serre
Harvard, Fall 1996

the subgroups $H^i_{nr}(k_v, A)$ are well defined. Let $P^i(k,A)$ be the subgroup of the product $\prod_{v\in V} H^i(k_v, A)$ consisting of the families (x_v) such that x_v belongs to $H^i_{nr}(k_v, A)$ for almost all $v \in V$. We have:

Proposition 21. *The canonical homomorphism* $H^i(k,A) \to \prod H^i(k_v, A)$ *maps* $H^i(k,A)$ *into* $P^i(k,A)$.

Indeed, every element x of $H^i(k,A)$ comes from an element $y \in H^i(L/k, A)$, where L/k is a suitable finite Galois extension. If T denotes the union of S and the set of places of k which are ramified in L, it is clear that the image x_v of x in $H^i(k_v, A)$ belongs to $H^i_{nr}(k_v, A)$ for all $v \notin T$, from which the proposition follows.

We shall denote by $f_i : H^i(k,A) \to P^i(k,A)$ the homomorphism defined by the preceding proposition. By prop. 18 of §5.5, we have:

$$P^0(k,A) = \prod H^0(k_v, A) \qquad \text{(direct product)},$$

$$P^2(k,A) = \coprod H^2(k_v, A) \qquad \text{(direct sum)}.$$

As for the group $P^1(k,A)$, Tate suggests denoting it by $\prod\!\!\!\!\coprod H^1(k_v, A)$, to emphasize that it is intermediate between a product and a sum.

The groups $P^i(k,A)$, $i \geq 3$, are simply the (finite) products of the $H^i(k_v, A)$, where v runs over the set of *real* archimedean places of k. In particular, we have $P^i(k,A) = 0$ for $i \geq 3$ if k is totally imaginary, or if A is of odd order.

Remark.

The map f_0 is obviously injective, and Tate has proved (cf. §6.3) that the f_i, $i \geq 3$, are bijective. In contrast, f_1 and f_2 are not necessarily injective (cf. Chap. III, §4.7).

Exercises.

1) Let w be an ultrametric place of the algebraic closure $\bar{k}$ of k. Show that the field $\bar{k}_w$ defined above is not complete [notice that it is a countable union of closed subspaces without interior point, and apply Baire's theorem]. Show that the completion of $\bar{k}_w$ is algebraically closed.

2) Define the $P^i(k,A)$ for negative i. Show that the system of $\{P^i(k,A)\}_{i\in\mathbf{Z}}$ forms a cohomological functor in A.

6.2 The finiteness theorem

The groups $P^i(k,A)$ defined in the preceding § can be given a natural *locally compact group topology* (a special case of the notion of a "restricted product" due to Braconnier): one takes as a neighborhood base of 0 the subgroups $\prod_{v\notin T} H^i_{nr}(k_v, A)$, where T runs over the set of finite subsets of V containing S. For $P^0(k,A) = \prod H^0(k_v, A)$, we get the *product* topology, which makes $P^0(k,A)$ a *compact* group. For $P^1(k,A) = \prod\!\!\!\!\coprod H^1(k_v, A)$ we get a locally compact group topology; for $P^2(k,A) = \coprod H^2(k_v, A)$, we get the *discrete* topology.

Theorem 7. *The canonical homomorphism*

$$f_i : H^i(k,A) \longrightarrow P^i(k,A)$$

is a proper map, when $H^i(k,A)$ *is given the discrete topology, and* $P^i(k,A)$ *the topology defined above* (i.e. the inverse image by f_i of a compact subset of $P^I(k,A)$ is *finite*).

We shall only prove this theorem for $i = 1$. The case $i = 0$ is trivial, and the case $i \geq 2$ follows from more precise results of Tate and Poitou which will be given in the next section.

Let T be a subset of V containing S, and let $P^1_T(k,A)$ be the subgroup $P_1(k,A)$ formed by the elements (x_v) such that $x_v \in H^1_{nr}(k_v, A)$ for all $v \notin T$. It is obvious that $P^1_T(k,A)$ is compact, and that conversely any compact subset of $P^1(k,A)$ is contained in one of the $P^1_T(k,A)$. It will therefore be enough to prove that the inverse image X_T of $P^1_T(k,A)$ in $H^1(k,A)$ is *finite*. By definition, an element $x \in H^1(k,A)$ belongs to X_T if and only if it is unramified outside T. Let us denote, as above, by K/k a finite Galois extension of k such that G_K acts trivially on A, and let T' be the set of places of K which extend the places of T. One can easily see that the image of X_T in $H^1(k,A)$ consists of the elements unramified outside T; since the kernel of $H^1(k,A) \to H^1(K,A)$ is finite, we are therefore led to showing that these elements are in finite number. So (up to replacing k with K), *we can assume that* G_k *acts trivially* on A. Therefore we have $H^1(k,A) = \mathrm{Hom}(G_k, A)$. If $\varphi \in \mathrm{Hom}(G_k, A)$, denote the extension of k corresponding to the kernel of φ by $k(\varphi)$; it is an abelian extension, and φ defines an isomorphism of the Galois group $\mathrm{Gal}(k(\varphi)/k)$ onto a subgroup of A. To say that φ is unramified outside T means that the extension $k(\varphi)/k$ is unramified outside T. Since the extensions $k(\varphi)$ are of bounded degree, the finiteness theorem we want is a consequence of the following more precise result:

Lemma 6. *Let* k *be an algebraic number field, and* r *an integer, and let* T *be a finite set of places of* k. *There exist only finitely many extensions of degree* r *of* k *which are unramified outside* T.

We reduce immediately to the case $k = \mathbf{Q}$. If E is an extension of $\mathbf{Q}$ of degree r unramified outside T, the discriminant d of E over $\mathbf{Q}$ is only divisible by prime numbers p belonging to T. In addition, the exponent of p in d is bounded (this follows, for instance, from the fact that there only exist a finite number of extensions of the local field $\mathbf{Q}_p$ which are of degree $\leq r$, cf. Chap. III, §4.2; see also [145], p. 67). Therefore there are only finitely many possible discriminants d. Since there exist only a finite number of number fields with a given discriminant (Hermite's theorem), this proves the lemma.

6.3 Statements of the theorems of Poitou and Tate

Retain the previous notations, and set $A' = \mathrm{Hom}(A, \mathbf{G}_m)$. The duality theorem for the local case, together with prop. 19 in §5.5, implies that $P^0(k,A)$ *is dual* *to* $P^2(k,A')$ *and* $P^1(k,A)$ *is dual to* $P^1(k,A')$ [one has to be careful with the archimedean places – this works because of the convention made at the start of §6.1.].

The following three theorems are more difficult. We just state them without proof:

Theorem A. *The kernel of* $f_1 : H^1(k,A) \to \prod H^1(k_v,A)$ *and the kernel of* $f_2' : H^2(k,A') \to \prod H^2(k_v,A')$ *are duals of each other.*

Note that this statement, applied to the module A', implies that the kernel of f_2 is finite; the case $i = 2$ of th. 7 follows immediately from that.

Theorem B. *For* $i \geq 3$, *the homomorphism*

$$f_i : H^i(k,A) \longrightarrow \prod H^i(k_v,A)$$

is an isomorphism.

[Of course, in the product, v runs through the real places of k, i.e. such that $k_v = \mathbf{R}$.]

Theorem C. *We have an exact sequence:*

$$\begin{array}{ccccccccc}
0 \to & H^0(k,A) & \to & \prod H^0(k_v,A) & \to & H^2(k,A')^* & \to & H^1(k,A) & \\
 & \text{(finite)} & & \text{(compact)} & & \text{(compact)} & & \text{(discrete)} & \searrow \\
 & & & & & & & & \prod H^1(k_v,A) \\
 & & & & & & & & \swarrow \text{(loc. compact)} \\
0 \leftarrow & H^0(k,A')^* & \leftarrow & \prod H^2(k_v,A) & \leftarrow & H^2(k,A) & \leftarrow & H^1(k,A')^* & \\
 & \text{(finite)} & & \text{(discrete)} & & \text{(discrete)} & & \text{(compact)} &
\end{array}$$

All the homomorphisms occurring in this sequence are continuous.

(Here, G^* is the dual – in Pontryagin's sense – of the locally compact group G.)

These theorems are given in Tate's Stockholm lecture [171], with brief hints of proofs. Other proofs, due to Poitou, can be found in the 1963 Lille Seminar, cf. [126]. See also Haberland [65] and Milne [116].

Bibliographic remarks for Chapter II

The situation is the same as for Chapter I: almost all the results are due to Tate. The only paper published by Tate on this subject is his Stockholm lecture [171], which contains lots of results (many more than it has been possible to discuss here), but very few proofs. Fortunately, the proofs in the local case were worked out by Lang [97]; and others can be found in a lecture by Douady at the Bourbaki Seminar [47].

Let us also mention:

1) The notion of "cohomological dimension" (for the Galois group G_k of a field k) was introduced for the first time by Grothendieck, in connection with his study of "Weil cohomology." Prop. 11 in §4.2 is due to him.

2) Poitou obtained the results of §6 at about the same time as Tate. He lectured on his proofs (which seem different from those of Tate) in the Lille Seminar [126].

3) Poitou and Tate were both influenced by the results of Cassels on the Galois cohomology of elliptic curves, cf. [26].