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Jürgen Neukirch

Algebraic Number Theory

Translated from the German
by Norbert Schappacher

With 16 Figures

Springer

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Foreword

It is a very sad moment for me to write this "Geleitwort" to the English translation of Jürgen Neukirch's book on Algebraic Number Theory. It would have been so much better, if he could have done this himself.

But it is also very difficult for me to write this "Geleitwort": The book contains Neukirch's Preface to the German edition. There he himself speaks about his intentions, the content of the book and his personal view of the subject. What else can be said?

It becomes clear from his Preface that Number Theory was Neukirch's favorite subject in mathematics. He was enthusiastic about it, and he was also able to implant this enthusiasm into the minds of his students.

He attracted them, they gathered around him in Regensburg. He told them that the subject and its beauty justified the highest effort and so they were always eager and motivated to discuss and to learn the newest developments in number theory and arithmetic algebraic geometry. I remember very well the many occasions when this equipe showed up in the meetings of the "Oberwolfach Arbeitsgemeinschaft" and demonstrated their strength (mathematically and on the soccer field).

During the meetings of the "Oberwolfach Arbeitsgemeinschaft" people come together to learn a subject which is not necessarily their own speciality. Always at the end, when the most difficult talks had to be delivered, the Regensburg crew took over. In the meantime many members of this team teach at German universities.

We find this charisma of Jürgen Neukirch in the book. It will be a motivating source for young students to study Algebraic Number Theory, and I am sure that it will attract many of them.

At Neukirch's funeral his daughter Christiane recited the poem which she often heard from her father: *Herr von Ribbeck auf Ribbeck im Havelland* by Theodor Fontane. It tells the story of a nobleman who always generously gives away the pears from his garden to the children. When he dies he asks for a pear to be put in his grave, so that later the children can pick the pears from the growing tree.

This is – I believe – a good way of thinking of Neukirch's book: There are seeds in it for a tree to grow from which the "children" can pick fruits in the time to come.

G. Harder

Chapter VI

Global Class Field Theory

§ 1. Idèles and Idèle Classes

The rôle held in local class field theory by the multiplicative group of the base field is taken in global class field theory by the idèle class group. The notion of idèle is a modification of the notion of ideal. It was introduced by the French mathematician *Claude Chevalley* (1909–1984) with a view to providing a suitable basis for the important local-to-global principle, i.e., for the principle which reduces problems concerning a number field K to analogous problems for the various completions $K_\mathfrak{p}$. *Chevalley* used the term "ideal element", which was abbreviated as id. el.

An **adèle** of K – this curious expression, which has the stress on the second syllable, is derived from the original term "additive idèle" – is a family

$$\alpha = (\alpha_\mathfrak{p})$$

of elements $\alpha_\mathfrak{p} \in K_\mathfrak{p}$ where $\mathfrak{p}$ runs through all primes of K, and $\alpha_\mathfrak{p}$ is integral in $K_\mathfrak{p}$ for almost all $\mathfrak{p}$. The adèles form a ring, which is denoted by

$$\mathbb{A}_K = \prod_\mathfrak{p} K_\mathfrak{p}.$$

Addition and multiplication are defined componentwise. This kind of product is called the "restricted product" of the $K_\mathfrak{p}$ with respect to the subrings $\mathcal{O}_\mathfrak{p} \subseteq K_\mathfrak{p}$.

The **idèle group** of K is defined to be the unit group

$$I_K = \mathbb{A}_K^*.$$

Thus an **idèle** is a family

$$\alpha = (\alpha_\mathfrak{p})$$

of elements $\alpha_\mathfrak{p} \in K_\mathfrak{p}^*$ where $\alpha_\mathfrak{p}$ is a unit in the ring $\mathcal{O}_\mathfrak{p}$ of integers of $K_\mathfrak{p}$, for almost all $\mathfrak{p}$. In analogy with $\mathbb{A}_K$, we write the idèle group as the restricted product

$$I_K = \prod_\mathfrak{p} K_\mathfrak{p}^*$$