Theta divisors and the Frobenius morphism

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Abstract

We introduce theta divisors for vector bundles and relate them to the ordinariness of curves in characteristic $p > 0$. We prove, following M. Raynaud, that the sheaf of locally exact differentials in characteristic $p > 0$ has a theta divisor, and that the generic curve in (any) genus $g \geq 2$ and (any) characteristic $p > 0$ has a cover that is not ordinary (and which we explicitely construct).

1 Theta divisors for vector bundles

Let k be an algebraically closed field and X a smooth proper connected curve over $\operatorname{Spec} k$ having genus g. We assume throughout that $g \geq 2$.

If E is a vector bundle (i.e. a locally free invertible sheaf) of rank r and degree d over X, we define its *slope* to be $\lambda = d/r$. The Riemann-Roch formula gives the Euler-Poincaré characteristic of E:

$$\chi(X, E) = h^0(X, E) - h^1(X, E) = r(\lambda - (g - 1))$$

In particular for $\lambda = g - 1$ (the *critical slope*) we have $\chi(X, E) = 0$; moreover, it is still true for any invertible sheaf L of degree 0 over X that $\chi(X, E \otimes L) = 0$, in other words, $h^0(X, E \otimes L) = h^1(X, E \otimes L)$. Under those circumstances, it is natural to ask the following question:

Question 1.1. *Suppose E has critical slope. Then for which invertible sheaves L of degree 0 (if any) is it true that $h^0(X, E \otimes L) = 0$ (and consequently also $h^1(X, E \otimes L) = 0$)? Is this true for some L, for many L, or for none?*

We start with a necessary condition. Suppose there were some subbundle $F \rightarrowtail E$ having slope $\lambda(F) > \lambda(E) = g - 1$. Then we would have

$\chi(X, F \otimes L) > 0$, hence $h^0(X, F \otimes L) > 0$. Now $H^0(X, F \otimes L) \rightarrowtail H^0(X, E \otimes L)$, so this implies $h^0(X, E \otimes L) > 0$. So if we are to have $h^0(X, E \otimes L) > 0$ for some L, this must not happen, and we say that E is *semi-stable*:

Definition 1.2. *A vector bundle E over X is said to be* stable *(resp.* semi-stable*) iff for every sub-vector-bundle F of E (other than* 0 *and E) we have $\lambda(F) < \lambda(E)$ (resp. $\lambda(F) \leq \lambda(E)$).*

Another remark we can make bearing some relation with question (1.1) is that, by the semicontinuity theorem, if we let L vary on the jacobian of X, the functions $h^0(X, E \otimes L)$ and $h^1(X, E \otimes L)$ are upper semicontinuous. This means that they increase on closed sets. In particular, if $h^0(X, E \otimes L) = 0$ (the smallest possible value) for some L, then this is true in a whole neighborhood of L, that is, for almost all L. We then say that this holds for a *general* invertible sheaf of degree 0 and we write $h^0(X, E \otimes L_{\mathrm{gen}}) = 0$.

Now introduce the jacobian variety J of X and let $\mathscr{L}$ be the (some) Poincaré sheaf (universal invertible sheaf of degree 0) on $X \times_{\operatorname{Spec} k} J$. We aim to use $\mathscr{L}$ to let L vary and provide universal analogues for our formulæ. Let

$$\begin{array}{c} X \times_{\operatorname{Spec} k} J \\ \big\downarrow {\scriptstyle f} \\ J \end{array}$$

be the second projection.

Consider the sheaf $E \otimes \mathscr{L}$ (by this we mean the twist by $\mathscr{L}$ of the pullback of E to $X \times_{\operatorname{Spec} k} J$). Our interest is mainly in the higher direct image $Rf_*(E \otimes \mathscr{L})$, which incorporates information about $H^i(X, E \otimes L)$ (and much more).

To be precise, we know that there exists a complex $0 \to M^0 \xrightarrow{u} M^1 \to 0$ of vector bundles on J that universally computes the $R^i f_*(E \otimes \mathscr{L})$, in the sense that the i-th cohomology group ($i = 0, 1$) of the complex is $R^i f_*(E \otimes \mathscr{L})$ and that this remains true after any base change $J' \to J$. (One particularly important such base change, of course, is the embedding of a closed point $\{L\}$ in J.)

Now M^0 and M^1 have the same rank, say s (because the Euler-Poincaré characteristic of E is 0). So we can consider the determinant of u, $\bigwedge^s M^0 \xrightarrow{\det u} \bigwedge^s M^1$, or rather

$$\mathcal{O}_J \xrightarrow{\det u} \bigwedge^s M^1 \otimes \left(\bigwedge^s M^0\right)^{\otimes -1}$$

Exactly one of the following two things happens:

- *Either* the determinant $\det u$ is zero (identically). In this case, u is nowhere (i.e. on no fiber) invertible, it always has a kernel and a cokernel: we have $h^0(X, E \otimes L) > 0$ (and of course $h^1(X, E \otimes L) > 0$) for all L.

- *Or* $\det u$ is a nonzero section of the invertible sheaf $\bigwedge^s M^1 \otimes \left(\bigwedge^s M^0\right)^{\otimes -1}$ on J and it defines a positive divisor θ_E on J, whose support is precisely the locus of L such that $h^0(X, E \otimes L) > 0$. We then have $h^0(X, E \otimes L_{\text{gen}}) = 0$.

In other words, precisely in the case where $h^0(X, E \otimes L_{\text{gen}}) = 0$ we can define a positive divisor θ_E on J which tells us "where the bundle E has cohomology". We call this divisor the *theta divisor* of the vector bundle E. And we will use the expression "to admit a theta divisor" as synonymous for $h^0(X, E \otimes L_{\text{gen}}) = 0$. For example, a vector bundle that admits a theta divisor is semi-stable (but the converse is not true, cf. [1]).

2 Enters the Frobenius morphism

We now assume that the base field k has characteristic $p > 0$. We then have a relative Frobenius morphism

$$\begin{tikzcd} X \arrow[rr, "\pi"] \arrow[dr] & & X_1 \arrow[dl] \\ & \operatorname{Spec} k & \end{tikzcd}$$

which is obtained by factoring the absolute Frobenius morphism through the pullback to X of the Frobenius on k (more descriptively, π has the effect, in projective space, of raising the coordinates to the p-th power, while X_1 is the curve obtained by raising to the p-th powers the coefficients in the equations defining X).

The curve X_1 has the same genus g as X. The morphism π is flat, finite and purely inseparable of degree p. From it we deduce a morphism $\mathcal{O}_{X_1} \to \pi_* \mathcal{O}_X$ (of $\mathcal{O}_{X_1}$-modules), which is mono because π is surjective. Call B_1 the cokernel, so that we have the following short exact sequence:

$$0 \to \mathcal{O}_{X_1} \to \pi_* \mathcal{O}_X \to B_1 \to 0 \tag{1}$$

Now the sheaf B_1 can be viewed in a different way: if we call d the differential $\mathcal{O}_X \to \Omega^1_X$ (between sheaves of $\mathbb{Z}$-modules) then $\pi_*(d)$ is $\mathcal{O}_{X_1}$-linear and has $\mathcal{O}_{X_1}$ as kernel. Consequently, B_1 can also be seen as the image of $\pi_*(d)$, hence its name of *sheaf of locally exact differentials*. As a subsheaf of the locally free sheaf $\pi_*\Omega^1_X$, it is itself a vector bundle.

In the short exact sequence (1) above, the vector bundles $\mathcal{O}_{X_1}$ and $\pi_*\mathcal{O}_X$ have rank 1 and p respectively, so that B_1 has rank $p-1$. On the other hand, since $\mathcal{O}_{X_1}$ and $\pi_*\mathcal{O}_X$ each have Euler-Poincaré characteristic $g-1$, we have $\chi(X_1, B_1) = 0$, or in other words, $\lambda(B_1) = g-1$ (the critical slope), and what we have said in the previous section applies to the sheaf B_1.

More precisely, we have the following long exact sequence in cohomology, derived from (1):

$$\begin{array}{ccccccccc} & & k & \xrightarrow{\sim} & k & & & & \\ & & \| & & \| & & & & \\ 0 & \to & H^0(X_1,\mathcal{O}_{X_1}) & \to & H^0(X,\mathcal{O}_X) & \to & H^0(X_1,B_1) & \to & \\ & \to & H^1(X_1,\mathcal{O}_{X_1}) & \to & H^1(X,\mathcal{O}_X) & \to & H^1(X_1,B_1) & \to & 0 \end{array}$$

Here the first arrow is an isomorphism as shown. Consequently, the arrow $H^1(X_1, \mathcal{O}_{X_1}) \to H^1(X, \mathcal{O}_X)$ is also an isomorphism if and only if $h^0(X_1, B_1) = 0$, or, what amounts to the same, $h^1(X_1, B_1) = 0$. This is again the same as saying that B_1 has a theta divisor (something which we will see is always true) and that it does not go through the origin.

Definition 2.1. *When the equivalent conditions mentioned in the previous paragraph are satisfied, we say that the curve X is* ordinary.

3 The sheaf of locally exact differentials has a theta divisor

In this section we prove the following result due to M. Raynaud ([1]):

Theorem 3.1. *If X is a smooth projective connected curve over an algebraically closed field k of characteristic $p > 0$ and B_1 is the sheaf of locally exact differentials on X_1, as introduced above, then we have $h^0(X_1, B_1 \otimes L_{\mathrm{gen}}) = 0$, i.e. the vector bundle B_1 admits a theta divisor (in particular, it is semi-stable).*

Thus we can state the fact that a curve is ordinary simply by saying that the theta divisor of B_1 does not go through the origin.

To start with, introduce the jacobians J and J_1 of X and X_1 respectively. Then J_1 is the Frobenius image of J, and we have a relative Frobenius morphism $F\colon J \to J_1$ that is purely inseparable of degree p^g; it corresponds to taking the norm on invertible sheaves of degree 0 — or, if we prefer using points, it takes $\mathcal{O}_X(\Sigma n_i x_i)$ to $\mathcal{O}_{X_1}(\Sigma n_i \pi(x_i))$. On the other hand, we also have the *Verschiebung* morphism in the other direction $V\colon J_1 \to J$, which corresponds to pulling back by π — or again, it takes $\mathcal{O}_{X_1}(\Sigma n_i x_i)$ to $\mathcal{O}_X(\Sigma p n_i \pi^{-1}(x_i))$. The Verschiebung map also has degree p^g. The composite of the Verschiebung and Frobenius morphisms, in any direction, is the raising to the p-th power.

We will show something more precise than just saying that B_1 has a theta divisor: we will actually show that this theta divisor does not contain all of $\ker V$ in the neighborhood of 0. However, we will see from actual equations that it "almost" does.

If L_1 is an invertible sheaf of degree 0 on X_1 (that is, a k-point of J_1), the short exact sequence (1) becomes, after tensoring by L_1:

$$0 \to L_1 \to \pi_* \pi^* L_1 \to B_1 \otimes L_1 \to 0 \tag{2}$$

Now let $\mathscr{L}_1$ be the Poincaré bundle on $X_1 \times_{\operatorname{Spec} k} J_1$. The universal analogue of (2) above is

$$0 \to \mathscr{L}_1 \to (\pi \times 1_{J_1})_* (\pi \times 1_{J_1})^* \mathscr{L}_1 \to B_1 \otimes \mathscr{L}_1 \to 0$$

But by the definition of the Verschiebung, the sheaf $(\pi \times 1_{J_1})^* \mathscr{L}_1$ is also $(1_X \times V)^* \mathscr{L}$ so that the exact sequence can be written as

$$0 \to \mathscr{L}_1 \to (\pi \times 1_{J_1})_* (1_X \times V)^* \mathscr{L} \to B_1 \otimes \mathscr{L}_1 \to 0$$

We now introduce projections as designated on the following diagram:

$$\begin{tikzcd}
X \times J \arrow[d, "f"'] & X \times J_1 \arrow[l, "1_X \times V"'] \arrow[r, "\pi \times 1_{J_1}"] \arrow[d, "g"'] \arrow[dl, phantom, "\square"] & X_1 \times J_1 \arrow[dl, "f_1"] \\
J & J_1 \arrow[l, "V"']
\end{tikzcd} \tag{3}$$

Now we want to calculate the $R(f_1)_*$ of this. For one thing, looking at the diagram (3) above, we see that $R(f_1)_* (\pi \times 1_{J_1})_* (1_X \times V)^* \mathscr{L}$ is $Rg_* (1_X \times V)^* \mathscr{L}$, and by base change (note that the morphism V is flat), this is $V^*Rf_*\mathscr{L}$. Thus we have the following distinguished triangle, in the derived category of the category of sheaves on J_1:

$$\begin{tikzcd} & R(f_1)_*(B_1 \otimes \mathscr{L}_1) \arrow[dl, "+1"'] & \\ R(f_1)_*\mathscr{L}_1 \arrow[rr, "a"] & & V^*Rf_*\mathscr{L} \arrow[ul] \end{tikzcd} \tag{4}$$

And the corresponding long exact sequence of cohomology is

$$0 \to (f_1)_*(B_1 \otimes \mathscr{L}_1) \to R^1(f_1)_*\mathscr{L}_1 \xrightarrow{a} V^*R^1f_*\mathscr{L} \to R^1(f_1)_*(B_1 \otimes \mathscr{L}_1) \to 0$$

(the two first terms cancel). We want to show that a is generically invertible.

To start with, consider a minimal resolution of $Rf_*\mathscr{L}$ in the neighborhood of the origin. It has the form

$$\mathcal{O}_{J,0} \xrightarrow{u'} \mathcal{O}_{J,0}^g$$

where $u'(1) = (x_1, \dots, x_g)$ is a system of parameters around 0. Indeed, this last statement is the same as saying, if u is the transpose of u', that the image of u is the maximal ideal of the regular local ring $\mathcal{O}_{J,0}$, and this is easy because $\{0\}$ is the largest closed subscheme of $\operatorname{Spec} \mathcal{O}_{J,0}$ on which $\mathscr{L}$ is trivial.

Now apply what we have just proven to J_1 on the one hand, and to J on the other, but pulling back by V, we find the following resolution for the arrow a in triangle (4):

$$\begin{tikzcd} R \arrow[r, "a_0"] \arrow[d, "u'"'] & R \arrow[d, "v'"] \\ R^g \arrow[r, "a_1"] & R^g \end{tikzcd} \tag{5}$$

where we have written $R = \mathcal{O}_{J_1,0}$, and where $u'(1) = (x_1, \dots, x_g)$ is a system of parameters of J_1 around 0 and $v'(1) = (y_1, \dots, y_g)$ is a regular sequence that gives an equation of $\ker V$ around 0. Now of course a_0 is just an element of R, and it is invertible because modulo the maximal ideal of R (that is, *at* the origin) the arrow a is just the identity on k. So we can assume that a_0 is the identity. What we want to prove is that $\det a_1$ is not zero (of course, it is invertible precisely when the curve is ordinary).

Consider the diagram (5) and its transpose (i.e. its image by the functor $\mathrm{Hom}_R(\cdot, R)$), and complete them both by adding the Koszul complex on either column. That is, consider the diagrams:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & \text{and} & 0 \arrow[d] & 0 \arrow[d] \\
R \arrow[r, "1"] \arrow[d, "u'"'] & R \arrow[d, "v'"] & & \wedge^g R^g \arrow[r, "\det a_1"] \arrow[d, dotted, no head] & \wedge^g R^g \arrow[d, dotted, no head] \\
R^g \arrow[r, "a_1"] \arrow[d, dotted, no head] & R^g \arrow[d, dotted, no head] & & R^g \arrow[r, "a_1^\vee"] \arrow[d, "v"'] & R^g \arrow[d, "u"] \\
\wedge^g R^g \arrow[r, "\det a_1"] \arrow[d] & \wedge^g R^g \arrow[d] & & R \arrow[r, "1"] \arrow[d] & R \arrow[d] \\
M' \arrow[r, "h'"] \arrow[d] & N' \arrow[d] & & N \arrow[r, "h"] \arrow[d] & M \arrow[d] \\
0 & 0 & & 0 & 0
\end{tikzcd}$$

in which we have written u, v and $a_1^\vee$ for the transposes of u', v' and a_1 respectively and M and N for the cokernels of u and v respectively. Since $u'(1)$ and $v'(1)$ are regular sequences, M and N are modules of finite length, and the Koszul complex is a resolution of them: the columns of both diagram are exact. We have $M' = \mathrm{Ext}^g_R(M, R)$ and $N' = \mathrm{Ext}^g_R(N, R)$ (since we have taken a resolution, transposed it, and shifted in g degrees). But since the Koszul complex is autodual (that is, the left column of the right diagram is the same as the right column of the left diagram, and *vice versa*), M and M' are the same and so are N and N'. Finally, it is known that (R being a regular local ring) the functor $\mathrm{Ext}^g_R(\cdot, R)$ is dualizing on modules of finite length. Now h is surjective as is seen on the diagram on the right, so that its image h' by the functor in question is injective. Hence $\det a_1$ is nonzero, what we wanted.

We can be more precise than this. As we have seen, M is isomorphic to k, and N to the local ring of $\ker V$ at 0: the support of θ_{B_1} swallows everything in $\ker V$ around the origin but just one k. (Incidentally, X is ordinary if and only if the support of θ_{B_1} does not contain the origin, so we recover the known fact that X is ordinary if and only if the local ring of $\ker V$ at the origin is k, i.e. V is étale.)

4 Constructing a non ordinary cover

We now present another result of M. Raynaud's ([2]), namely the fact that a finite étale cover of an ordinary curve is not necessarily ordinary, even when the base curve is generic. In fact, we obtain a cover $Y \twoheadrightarrow X$ such that the image of the map $J(X_1) \rightarrowtail J(Y_1)$ on the jacobians is completely contained in the support of the theta divisor of B_1 on Y_1 — and in particular 0 is, so that Y is not ordinary. The construction is sufficiently general to apply to the generic curve (for a given genus $g \geq 2$ and characteristic p). We also get estimations on the Galois group of Y over X; a theorem of Nakajima states that an abelian cover of the generic curve is ordinary, so we have to work with non abelian groups if we want a non ordinary cover — however, we will see that a nilpotent group can suffice.

We start with a few generalities on representations of the fundamental group of curves. We refer to [2] for details. If $\rho\colon \pi_1(X) \to GL(r,k)$ is a representation of the fundamental group of X in k-vector spaces of rank r, and ρ has open kernel (or, which amounts to the same, ρ is continuous and has finite image), then ρ defines a locally constant étale sheaf in k-vector spaces of rank r on X, written $\mathbb{V}_\rho$ (very succintly, $\mathbb{V}_\rho$ can be obtained as follows: find a Galois cover $Y \twoheadrightarrow X$ whose Galois group factors through the kernel of ρ, then make $\pi_1(X)/\ker\rho$ act on $Y \times k^r$ componentwise, and take the fixed points of that action). Tensoring $\mathbb{V}_\rho$ by $\mathcal{O}_X$ gives a Zarisky sheaf V_ρ which is locally free of rank r and has degree 0, i.e. a vector bundle of rank r and slope 0 on X. Among the functoriality properties of V_ρ cited in [2], we will need the fact that if $Y \overset{a}{\twoheadrightarrow} X$ is finite étale and ρ is a representation of $\pi_1(Y)$ as above then a_*V_ρ is precisely $V_{\rho'}$, where ρ' is the representation of $\pi_1(X)$ induced by ρ.

We say that a representation ρ as above has a theta divisor (respectively, is ordinary) if and only if the sheaf $V_{1,\rho} \otimes B_1$ on X_1 has a theta divisor (respectively, has a theta divisor that does not go through the origin), $V_{1,\rho}$ being the bundle V_ρ as above constructed on X_1. Thus, we have seen that the trivial representation has a theta divisor, and it is ordinary precisely when the curve X is ordinary. The existence of a theta divisor for B shows that if L is a general invertible sheaf of finite order n prime to p then the representation ρ of rank 1 associated to it is ordinary.

Theorem 4.1. *Let k be an algebraically closed field of characteristic $p > 0$, and let X be the generic curve of any genus $g \geq 2$ over* $\operatorname{Spec} k$. *Then there* *exists a Galois cover of X with solvable Galois group of order prime to p that is not ordinary.*

Let X be as stated, and let J be its jacobian. If we choose a base point on X then we get a map $S^{g-1}X \to J$ from the $(g-1)$-th symmetric power of X, whose image defines a positive divisor on J, called the *classical theta divisor*, and written Θ. Let $N = \mathcal{O}_J(\Theta)$ be the invertible sheaf defined by Θ. We can assume that N is symmetric (i.e. that if $\iota\colon J \to J$ is the inverse map then $\iota^* N = N$) and we will do so.

Let n be a positive integer that is prime to p, and denote by α the multiplication by n map on J, which is étale of degree n^{2g}. Call A the kernel of α, the (étale) set of points of J whose order divides n. Because we have chosen N symmetric, we have $\alpha^* N = N^{\otimes n^2}$.

We recall (cf. [3]) that the kernel $H(N^{\otimes n})$ is the subgroup of closed x in J such that $T_x^* N^{\otimes n} \cong N^{\otimes n}$ (where T_x denotes translation by x). This kernel is obviously A. Now in [3], D. Mumford defines another, more interesting, group associated to an invertible sheaf on an abelian variety. In our case, it is the group

$$\mathscr{G}(N^{\otimes n}) = \{(x,\varphi) \mid \varphi\colon N^{\otimes n} \tilde{\to} T_x^* N^{\otimes n}\}$$

with multiplication defined in the obvious way. There is a short exact sequence

$$1 \to k^\times \to \mathscr{G}(N^{\otimes n}) \to H(N^{\otimes n}) \to 1$$

and in fact $k^\times$ is precisely the center of $\mathscr{G}(N^{\otimes n})$. The commutator of two elements of $\mathscr{G}(N^{\otimes n})$ is an element of $k^\times$ and it depends only on the class in $H(N^{\otimes n})$ of the two elements. Thus, the commutator defines a skew-symmetric biadditive form $\langle\cdot,\cdot\rangle\colon A \times A \to k^\times$. It is moreover shown in [3] that this form is non degenerate.

Let B a maximal totally isotropic subgroup of A for the form we have just defined. So B has order n^g, and $C = A/B$ has order n^g. We factor α as follows

$$\begin{tikzcd} J \arrow[dd, "\alpha"'] \arrow[dr, "\beta"] & \\ & J' = J/B \arrow[dl, "\gamma"] \\ J & \end{tikzcd}$$

where β has kernel B and γ has kernel (identified with) C.

Because B is isotropic, by a result in [3], the sheaf $N^{\otimes n}$ descends to an invertible sheaf M on J', i.e. a sheaf such that $N^{\otimes n} = \beta^* M$. And M is a principal polarization on J'. Now note that $\gamma^* N$ and $M^{\otimes n}$ have the same pullback (namely $N^{\otimes n^2}$) by β; if n is odd we can choose M to be symmetric, so that $\gamma^* N$ and $M^{\otimes n}$ coincide. We will now suppose this to be the case.

If L is an invertible sheaf on J that is algebraically equivalent to 0 (that is, a closed point of $J^\vee$) then we have $\gamma_*(M \otimes \gamma^* L) \cong (\gamma_* M) \otimes L$, so that $h^0(J', M \otimes \gamma^* L) = h^0(J, (\gamma_* M) \otimes L)$. Now the point is that $M \otimes \gamma^* L$ is a principal polarization on J, so this number is 1. In particular $h^0(J, (\gamma_* M) \otimes L) > 0$, and this implies that for any invertible sheaf L of degree 0 on X we have $h^0(X, F \otimes L) > 0$, where F is the restriction of $\gamma_* M$ to X. This is a good first step, but we need to twist F by an invertible sheaf having the right degree to compensate for the slope of F (since the sheaves V_ρ have slope zero).

We now calculate the slope of F. Its rank is n^g. Introduce the curves Y and Z that are inverse image of X by γ and α respectively, thus:

$$\begin{tikzcd} Z \arrow[dd, "\alpha"'] \arrow[dr, "\beta"] & \\ & Y \arrow[dl, "\gamma"] \\ X & \end{tikzcd}$$

The degree of N restricted to X is well-known: it is g. Pulling this back by α, we see that the degree of $N^{\otimes n^2}$ restricted to Z is gn^{2g}, and that of $N^{\otimes n}|Z$ is gn^{2g-1}. Descending to Y, we see that the degree of $M|Y$ is gn^{g-1}. So the slope of F is finally g/n.

Now assume that g divides n, i.e. that $g/n = d$, the slope of F, is an integer. The degree of $N|X$ is $g = nd$, so there exists an invertible sheaf P of degree d on X such that $N|X = P^{\otimes n}$. Let $L' = (M|Y) \otimes \gamma^* P^{\otimes -1}$, which is an invertible sheaf of degree zero. Its inverse image $L'' = \beta^* L'$ is such that $L''^{\otimes n}$ is trivial on Z, so that the order of L'' divides n (in fact, it is exactly n, but we won't need this). If $E = \gamma_* L'$ then $E = F \otimes P^{\otimes -1}$, which is an invertible sheaf of degree 0 on X and satisfies $h^0(X, E \otimes L_{d,\mathrm{gen}}) > 0$ for a general invertible sheaf $L_{d,\mathrm{gen}}$ of degree d on X.

Now L'' is of order dividing n, so there is a cyclic covering of degree n $Z'' \twoheadrightarrow Z$ which trivializes it. It is Z'' that we will prove not to be ordinary (under certain numerical conditions at least). The invertible sheaf L' of degree 0 corresponds to an abelian representation of $\pi_1(Y)$ that factors trough $\pi_1(Z'')$, and when we induce that representation to $\pi_1(X)$ we see that $E = \gamma_* L'$ is of the form V_ρ for some representation ρ of $\pi_1(X)$ that factors through $\pi_1(Z'')$ (and which can actually be described: see [2]).

All these constructions were performed on X and J. They could equally well have been performed on X_1 and J_1. We now consider E as a sheaf of X_1. We have seen $h^0(X, E \otimes L_{d,\mathrm{gen}}) > 0$ and we wish to have $h^0(X, E \otimes B_1 \otimes L_{\mathrm{gen}}) > 0$. We are therefore done if we can show that B_1 contains an invertible subsheaf of degree d.

But A. Hirschowitz claims in [4] and proves in [5] that a general bundle of rank r_0 and slope λ_0 contains a subbundle of rank r' and slope λ' (the quotient having rank $r'' = r_0 - r'$ and slope λ'') if $\lambda'' - \lambda' \geq g - 1$. If we are looking for $r' = 1$ and $\lambda' = d$, with $r_0 = p - 1$ and $\lambda_0 = g - 1$ (the numerical values of B_1), so $r'' = p - 2$ and $\lambda'' = [(g-1)(p-1) - d]/(p-2)$, this condition is satisfied iff $(g - 1 - d)(p-1)/(p-2) \geq g - 1$, that is iff $d \leq \frac{g-1}{p-1}$. By deforming and specializing to B_1, we see that if this inequality is satisfied then B_1 contains an invertible sheaf of degree d.

Finally, we have shown that if p and g are such that there exists a positive odd integer n, prime to p, dividing g, and satisfying $\frac{g}{n} \leq \frac{g-1}{p-1}$ then the generic curve X of genus g in characteristic p has a covering that is not ordinary. This is not always the case, but we can always reduce to that case by first taking a cyclic cover X' of degree m prime to p of X (X' then has genus $g' = 1 + m(g-1)$), and apply the result to X'. Here are the details:

- If p is odd, take m even, not multiple of p and large enough so that $g' = 1 + m(g-1) \geq p$.
  - If p does not divide g', then $n = g'$ works (it is odd because m is even, it is prime to p, and $\frac{g'}{n} = 1 \leq \frac{g'-1}{p-1}$ because $g' \geq p$).
  - If p does divide g' then we double m and this is no longer the case, so we are reduced to the previous point.
- If $p = 2$, write $g = 2^r s$ with s odd.
  - If $s \geq 3$, take $m = 1$, $n = s$. (Then n is odd, and $\frac{g}{n} = 2^r \leq g - 1$.)
  - If $s = 1$ then $g = 2^r$.
    - If $r \geq 2$, take $m = 3$, $n = g'/2 = 3 \times 2^{r-1} - 1$. (Then n is odd, and $\frac{g'}{n} = 2 \leq g' - 1$.)

$*$ If $r = 1$ so $g = 2$ and we take $m = 5$, $g' = 6$, $n = 3$.

Finally, we note that our final covering was constructed as a composite $Z'' \twoheadrightarrow Y \twoheadrightarrow X' \twoheadrightarrow X$ of coverings all of which are abelian: it is therefore solvable.

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