On a few points concerning the philosophy of mathematics

The Quest for Truth

The notion of Truth is certainly one of the most important ones in mathematics. Indeed, at a sufficiently abstract level, one may say that the role of mathematics is to tell us which ones are true among such statements as, say:

"$\forall n \forall x \forall y \forall z ((n \geq 3 \wedge x^n + y^n = z^n) \Longrightarrow xyz = 0)$"

"*All the zeroes of the Riemann zeta-function lie on the critical axis*"

"*Any compact orientable smooth manifold of dimension* 3 *that has the homotopy type of the* 3*-sphere is homeomorphic to the* 3*-sphere*"

(It happens that, at the time of writing, the first is known to be true whereas the situation of the other two is unknown.)

Now what does it mean to be true? That is where things become difficult. In the first place, to a logician studying a formal system defined from some axioms, there is no such thing as "absolute" truth: truth is always relative to some model (in which the axioms are satisfied), and the only absolute thing is provability. A statement is said to be provable when it can be derived from the axioms by means of well-defined rules, and Gödel's completeness theorem assures us that the provable statements are precisely those which are true in every model. In our case, the formal system can be chosen as Zermelo-Fraenkel set theory (ZFC, with axiom of choice), and the axioms are extensionality, null set, pairing, union, power set, selection, replacement, foundation and choice. So one might reject the notion of truth entirely and limit the role of mathematics to finding which statements are provable in ZFC.

In this case, a statement such as "*Every uncountable set of reals can be put in bijection with the set of all reals*" becomes no longer true or untrue but merely unprovable (as well as its converse), and the whole problem seems to have melted away: to know whether the continuum hypothesis is true or not is merely a question of knowing what exactly is meant by a "set". To give an analogous situation, in the theory of groups (whose axioms are "$\forall x \forall y \forall z (x(yz) = (xy)z)$", "$\forall x (x1 = x \wedge 1x = x)$" and "$\forall x \exists y (xy = 1 \wedge yx = 1)$") the statement "$\forall x \forall y (xy = yx)$" is undecidable (precisely because some groups are commutative and others aren't). Asking whether it is true is as meaningless as asking "Is the Group Commutative?": it depends which group one is talking about. So one may, correspondingly, say that the truth of the continuum hypothesis depends merely on what is meant by a "set". And the ZFC axioms themselves are not truths but merely (attempts at) a definition of what sets are by how they should behave. The fact that our "definition" of sets is insufficient to resolve every property is an inevitable consequence of Gödel's second incompleteness theorem.

This is an extremely formalist point of view. What it means is that since sets do not exist in nature, no experiment can be done on them, so that talking about the truth or falsehood of a statement on sets (in particular whether every uncountable set of reals can be put in bijection with the set of all reals) is as absurd as asking whether dragons are green or whether vampires dislike onion as much as garlic. In the case of vampires, one might take Bram Stoker's *Dracula* as a set of axioms, and the above statement would be undecidable.

But is this situation entirely satisfactory? It does have one major problem, and that is that the statement "*A is provable*" is itself a mathematical statement, precisely one of formal arithmetic, and one is again trying to determine whether it is true. But it may also be undecidable. In fact, even if A is the statement "$0 = 1$", the statement "*A is provable*" is the statement "ZFC is inconsistent", and it is not refutable under Peano's axioms (and presumably not provable either, though *that* is also not provable). Well, no matter, one can just add Consis ZFC as an axiom (of arithmetic!), and this is exactly what permits us to prove that $0 = 1$ is unprovable. Moreover, results by Gödel and Cohen then allow us to prove that the continuum hypothesis is not provable and neither is its negation. So have we progressed? Not really, because a "stronger" statement like the consistency of an inaccessible cardinal (that is, the consistency of ZFC plus the axiom that there exists an inaccessible cardinal) remains unprovable and (presumably) irrefutable. And in fact all the Large Cardinal Axioms of set theory (the existence of inaccessible, respectively Mahlo, respectively measurable, respectively supercompact, cardinals) give rise to corresponding statements of *arithmetic* which are just as troublesome (the consistency of inaccessible, respectively Mahlo, respectively measurable, respectively supercompact, cardinals).

Of course, we may take these various consistency statements as axioms to add to Peano's axioms — albeit with the certitude that no number of them can every be satisfactory. It is perhaps morally comforting to know that if we start with PA_0 being Peano's axioms and by ordinal induction we define PA_α to be the formal system obtained by adding one statement (if there is one) that is not decided by the formal systems PA_β for $\beta < \alpha$, then in this way we eventually decide every statement of arithmetic, before even we reach ω_1; it is perhaps morally comforting, but it is a trivial fact, and besides it is unusable for any practical (or even epistemological) purpose.

An interesting alternative would be to take as axioms not the consistency of the large cardinal axioms but rather their *in*consistency (which really means, just the inconsistency of inaccessible cardinals). At least, this axiom is known to be consistent (repeat that: it is known that it is consistent to suppose that it is inconsistent to suppose that there are inaccessible cardinals; whereas it is not known, and cannot be shown, that it is consistent to suppose that it is consistent to suppose that there are inaccessible cardinals; get that?). Still, this axiom is very unsatisfactory in that it tells us that there is a proof that there are no inaccessible cardinals — one would much prefer actually having such a proof than an axiom that tells us it exists. And what if the proof doesn't exist? Ah, but one is talking truth again! The point is that a model for such an axiom is a nonstandard model, in which there are some "very large" integers, and a "very long" proof that inaccessible cardinals are inconsistent. In fact, it is even consistent with Peano's axioms to assume that Peano's axioms are inconsistent (that is Gödel's incompleteness theorem), so that *there is a model of Peano's axioms in which there exists a proof that Peano's axioms are inconsistent* (at least, provided that Peano's axioms actually are consistent). To state it more brutally: mathematical reasoning breaks down on very long (nonstandard) proof lengths. Wow!

So let us recapitulate: having previously proposed that mathematical truths on sets do not have any meaning, but only their provability, we are now faced the problem of truths on integers. One can go on saying that they also do not have any meaning, and that therefore even provability does not have any meaning, but only provability respective to such or such a model of Peano's axioms. Anyway, for all intents and purposes, one is not concerned with all natural numbers but only with the first few of them, say at most 10^{1000} — no proof we can ever see will get longer than that. So a strict finitist would just reject any notion of truth and just consider valid a statement for which we have a proof — the question of knowing whether there exists a proof then loses meaning, and only remains the, well-defined, question of whether such-or-such is indeed a proof for a given statement. I do not believe that this is a tenable position. There is something in the universality of natural numbers that forces me to believe that they are truly "natural", that they are more than an illusion. In other words, something forces me to believe that whereas there is no "natural" model for set theory (just as, and everyone will admit that, there is none for the theory of groups as mentioned earlier), there is one for Peano's axioms. This idea is, I believe, somewhat implicit in this famous phrase by Leopold Kronecker:

"God made the integers; all else is the work of Man."

At the very least, I think it is unreasonable to dispute that Π^1 and Σ^1 sentences over the naturals have a meaning. A Π^1 sentence is one which can be written $\forall n P(n)$ where $P(n)$ is a Σ^0 sentence, that is, one which has only bounded quantifiers; and a Σ^1 sentence is one whose negation is Π^1 (in other words, it can be written $\exists n P(n)$ for some Σ^0 sentence $P(n)$). The point is that a Σ^0 sentence can be decided in a deterministic way, in other words, there is a *primitive recursive* function which takes any Σ^0 sentence over the integers to its truth value (just test all the cases, and there is a finite number of them since all quantifiers are bounded). In other words, the set of true Σ^0 sentences is recursive. That is not the case for Σ^1 sentences, but at least the set of true Σ^1 sentences is recursively enumerable. So it is possible, at least with a *gedankenexperiment*, to decide a Σ^1 sentence: to decide $\exists n P(n)$, start a Turing machine that will successively decide $P(0), P(1), \ldots$. If the sentence is true, the machine will stop, otherwise it will go on forever. This is not a practical test, but I find it difficult to admit that the statement "the machine stops" has *no* truth value.

In fact, questions about the nature of our physical universe are relevant in this question. Indeed, if the universe is infinite in time, then the gedankenexperiment mentioned earlier takes a well-defined meaning, so Σ^1 statements about the integers *must* have a well-defined truth value. In fact, if the universe contains, anyhow, anywhere, a kind of infinity (whether in time, in space, in the possibility of subdividing space or time, or in virtual energy that may be created, or in any other kind of way), then it contains a "copy" of the set of natural numbers, so that natural numbers acquire a physical reality, and therefore statements about them acquire have a meaningful truth value (at any rate, Σ^1 or Π^1 statements).