Examples in number theory

Gaussian integers.

We consider the field extension $K/k := \mathbb{Q}(i)/\mathbb{Q}$. This extension is Galois with Galois group $\{1, \sigma\}$, satisfying $\sigma^2 = 1$, where σ is complex conjugation ($\sigma(i) = -i$). A basis of K over k is given by $1, i$; the trace is $\mathrm{Tr}(x+iy) = 2x$ and the norm is $N(x+iy) = x^2+y^2$; the degree of the extension is 2. It is straightforward to check that the ring $\mathfrak{o}_K$ of integers of K is $\mathbb{Z}[i]$. This ring is easily seen to be principal; in fact, it is even euclidian. The roots are unity are $\{1, i, -1, -i\}$, and the units are $\{1, i, -1, -i\}$.

The (global) different is easily seen to be $\mathfrak{D} = (2) = (1+i)^2$, and the discriminant is -4.

We now consider a few particular places.

1. The prime 2.

Let $\mathfrak{p} = (2)$ and v be the corresponding place. The only prime of $\mathbb{Z}[i]$ lying above $\mathfrak{p}$ is $\mathfrak{P} = (1+i) = (1-i)$. Let w be the corresponding place. We have $\mathfrak{p}\mathfrak{o}_K = \mathfrak{P}^2$: in other words, the prime 2 ramifies in $\mathbb{Z}[i]$, and the ramification index is $e = 2$; the prime $\mathfrak{P}$ is totally and strongly ramified. The residue field is $\mathbb{Z}[i]/\mathfrak{P} = \mathbb{F}_2$, which is equal to $\mathbb{Z}/\mathfrak{p}$: in other words, the modular degree is $f = 1$. The local degree is $ef = 2$, which means that the local field $K_w = \mathbb{Q}(i)_w$ is of degree 2 over $k_v = \mathbb{Q}_v = \mathbb{Q}_2$. Indeed, $\mathbb{Q}(i)_w = \mathbb{Q}_2(i)$. The local trace and norm extend the global trace and norm; in fact, $\mathrm{Tr}_w(x+iy) = 2x$ and $N_w(x+iy) = x^2+y^2$. The decomposition group $G_{\mathfrak{P}}$ of the prime $\mathfrak{P}$ is $G = \{1, \sigma\}$; the galois group $\bar{G}_{\mathfrak{P}}$ of the residue field extension $\mathbb{F}_2/\mathbb{F}_2$ is $\{1\}$. The inertia group is therefore $G = \{1, \sigma\}$. The decomposition field is $\mathbb{Q}$, and the inertia field is also $\mathbb{Q}$. The local different (that is, the $\mathfrak{P}$-component of the different) is $\mathfrak{D}_{\mathfrak{P}} = \mathfrak{P}^2$, witnessing again the fact that the prime in question is strongly ramified. Writing $|\cdot|_w$ for the absolute value on K_w which extends $|\cdot|_v = |\cdot|_2$, and $\|\cdot\|_w$ for the normalized absolute value on K_w (the module), we have $|2|_w = \frac{1}{2}$, $|1+i|_w = \frac{1}{\sqrt{2}}$, $|i|_w = 1$, and $\|2\|_w = \frac{1}{4}$, $\|1+i\|_w = \frac{1}{2}$, $\|i\|_w = 1$; in fact, $\|\cdot\|_w = |\cdot|_w^2$ because 2 is the local degree.

2. The prime 3.

Let $\mathfrak{p} = (3)$ and v be the corresponding place. The only prime of $\mathbb{Z}[i]$ lying above $\mathfrak{p}$ is $\mathfrak{P} = (3)$. Let w be the corresponding place. We have $\mathfrak{p}\mathfrak{o}_K = \mathfrak{P}$: in other words, the prime 3 does not ramify in $\mathbb{Z}[i]$, that is, the ramification index is $e = 1$. The residue field is $\mathbb{Z}[i]/\mathfrak{P} = \mathbb{F}_9$, which is quadratic over $\mathbb{Z}/\mathfrak{p}$: in other words, the modular degree is $f = 2$. The local degree is $ef = 2$, which means that the local field $K_w = \mathbb{Q}(i)_w$ is of degree 2 over $k_v = \mathbb{Q}_v = \mathbb{Q}_3$. Indeed, $\mathbb{Q}(i)_w = \mathbb{Q}_3(i)$. The local trace and norm extend the global trace and norm; in fact, $\mathrm{Tr}_w(x+iy) = 2x$ and $N_w(x+iy) = x^2+y^2$. The decomposition group $G_{\mathfrak{P}}$ of the prime $\mathfrak{P}$ is $G = \{1, \sigma\}$; the galois group $\bar{G}_{\mathfrak{P}}$ of the residue field extension $\mathbb{F}_9/\mathbb{F}_3$ is $\{1, \bar{\sigma}\}$. The inertia group is therefore $\{1\}$. The decomposition field is $\mathbb{Q}$, and the inertia field is $\mathbb{Q}(i)$. The local different (that is, the $\mathfrak{P}$-component of the different) is $\mathfrak{D}_{\mathfrak{P}} = (1)$, witnessing again the fact that the prime in question is unramified. Writing $|\cdot|_w$ for the absolute value on K_w which extends $|\cdot|_v = |\cdot|_3$, and $\|\cdot\|_w$ for the normalized absolute value on K_w (the module), we have $|3|_w = \frac{1}{3}$, $|i|_w = 1$, and $\|3\|_w = \frac{1}{9}$, $\|i\|_w = 1$; in fact, $\|\cdot\|_w = |\cdot|_w^2$ because 2 is the local degree.

3. The prime 5.

Let $\mathfrak{p} = (5)$ and v be the corresponding place. There are two primes of $\mathbb{Z}[i]$ lying above $\mathfrak{p}$, namely $\mathfrak{P} = (1+2i)$, and $\mathfrak{P}' = (1-2i)$. Let w, w' be the corresponding places. We have $\mathfrak{p}\mathfrak{o}_K = \mathfrak{P}\mathfrak{P}'$: in other words, the prime 5 splits in $\mathbb{Z}[i]$. Since neither prime ramifies, the two ramification indices are $e = e' = 1$. The residue fields are $\mathbb{Z}[i]/\mathfrak{P} = \mathbb{F}_5$ and $\mathbb{Z}[i]/\mathfrak{P}' = \mathbb{F}_5$, which are both equal to $\mathbb{Z}/\mathfrak{p}$: in other words, the modular degrees are $f = f' = 1$. Note that though the two residue fields are equal, the canonical maps from $\mathbb{Z}[i]$ to one and the other are *not* equal: in $\mathbb{Z}[i]/\mathfrak{P}$, we have $i = 2$, whereas in $\mathbb{Z}[i]/\mathfrak{P}'$ we have $i = 3$. The equality of the ramification indices and the modular degrees was necessary, since the extension K/k is Galois, so the Galois group operates transitively on the primes lying above $\mathfrak{p}$; in this case, $\sigma\mathfrak{P} = \mathfrak{P}'$ and $\sigma\mathfrak{P}' = \mathfrak{P}$. The local degrees are $ef = e'f' = 1$, which means that the local fields $K_w = \mathbb{Q}_w$ and $K_{w'} = \mathbb{Q}_{w'}$ are equal to $k_v = \mathbb{Q}_v = \mathbb{Q}_5$. Indeed, the polynomial X^2+1 is split over $\mathbb{Q}_5$ as $X^2+1 = (X - ...212)(X - ...233)$. Note that though the two local fields are equal, the canonical embeddings from $\mathbb{Z}[i]$ to one and the other are *not* equal. Both the local trace and the local norm are the identity on $K_w = K_{w'} = k_v$; however, they are not equal on K (since the embedding of K in K_w is not equal to that of K in $K_{w'}$). The decomposition group $G_{\mathfrak{P}}$ of the prime $\mathfrak{P}$ is $G = \{1\}$; the galois group $\bar{G}_{\mathfrak{P}}$ of the residue field extension $\mathbb{F}_5/\mathbb{F}_5$ is $\{1\}$. The inertia group is therefore $\{1\}$. The decomposition field is $\mathbb{Q}(i)$, and the inertia field is also $\mathbb{Q}(i)$. The local different (that is, the $\mathfrak{P}$-component of the different) is $\mathfrak{D}_{\mathfrak{P}} = (1)$, witnessing again the fact that the prime in question is unramified. Writing $|\cdot|_w$ for the absolute value on K_w which extends $|\cdot|_v = |\cdot|_5$, and $\|\cdot\|_w$ for the normalized absolute value on K_w (the module), we have $|5|_w = \frac{1}{5}$, $|1+2i|_w = \frac{1}{5}$, $|1-2i|_w = 1$, $|i|_w = 1$, and $\|5\|_w = \frac{1}{5}$, $\|1+2i\|_w = \frac{1}{5}$, $\|1-2i\|_w = 1$, $\|i\|_w = 1$; in fact, $\|\cdot\|_w = |\cdot|_w$ because 1 is the local degree. (Of course, all the final discussion applies, *mutatis mutandis*, to w' and $\mathfrak{P}'$ instead of w and $\mathfrak{P}$).

4. The infinite place.

Let $v = \infty$ be the infinite place of $\mathbb{Q}$ (it is real). The only place of $\mathbb{Q}(i)$ lying above v is a complex place w. The local degree is 2, which means that the local field $K_w = \mathbb{Q}(i)_w$ is of degree 2 over $k_v = \mathbb{Q}_v = \mathbb{R}$. Indeed, $\mathbb{Q}(i)_w = \mathbb{C}$. The local trace and norm extend the global trace and norm; in fact, $\mathrm{Tr}_w(x+iy) = 2x$ and $N_w(x+iy) = x^2+y^2$. Writing $|\cdot|_w$ for the absolute value on K_w which extends $|\cdot|_v = |\cdot|$, and $\|\cdot\|_w$ for the normalized absolute value on K_w (the module), we have $|2|_w = 2$, $|1+i|_w = \sqrt{2}$, $|i|_w = 1$, and $\|2\|_w = 4$, $\|1+i\|_w = 2$, $\|i\|_w = 1$; in fact, $\|\cdot\|_w = |\cdot|_w^2$ because 2 is the local degree.

Witnessing the product formula, we have, e.g.:

$$\begin{aligned}&\|i\|_{1+i} \times \|i\|_3 \times \|i\|_{1+2i} \times \|i\|_{1-2i} \times \cdots \times \|i\|_\infty \\ =&1 \times 1 \times 1 \times 1 \times \cdots \times 1 = 1\end{aligned}$$

$$\begin{aligned}&\|2\|_{1+i} \times \|2\|_3 \times \|2\|_{1+2i} \times \|2\|_{1-2i} \times \cdots \times \|2\|_\infty \\ =&\frac{1}{4} \times 1 \times 1 \times 1 \times \cdots \times 4 = 1\end{aligned}$$

$$\begin{aligned}&\|1+i\|_{1+i} \times \|1+i\|_3 \times \|1+i\|_{1+2i} \times \|1+i\|_{1-2i} \times \cdots \times \|1+i\|_\infty \\ =&\frac{1}{2} \times 1 \times 1 \times 1 \times \cdots \times 2 = 1\end{aligned}$$

$$\begin{aligned}&\|3\|_{1+i} \times \|3\|_3 \times \|3\|_{1+2i} \times \|3\|_{1-2i} \times \cdots \times \|3\|_\infty \\ =&1 \times \frac{1}{9} \times 1 \times 1 \times \cdots \times 9 = 1\end{aligned}$$

$$\begin{aligned}&\|5\|_{1+i} \times \|5\|_3 \times \|5\|_{1+2i} \times \|5\|_{1-2i} \times \cdots \times \|5\|_\infty \\ =&1 \times 1 \times \frac{1}{5} \times \frac{1}{5} \times \cdots \times 25 = 1\end{aligned}$$

$$\begin{aligned}&\|1+2i\|_{1+i} \times \|1+2i\|_3 \times \|1+2i\|_{1+2i} \times \|1+2i\|_{1-2i} \times \cdots \times \|1+2i\|_\infty \\ =&1 \times 1 \times \frac{1}{5} \times 1 \times \cdots \times 5 = 1\end{aligned}$$

Square root of 2.

We consider the field extension $K/k := \mathbb{Q}(\sqrt{2})/\mathbb{Q}$. This extension is Galois with Galois group $\{1, \sigma\}$, satisfying $\sigma^2 = 1$, where σ is defined by $\sigma(\sqrt{2}) = -\sqrt{2}$. A basis of K over k is given by $1, \sqrt{2}$; the trace is $\mathrm{Tr}(x + y\sqrt{2}) = 2x$ and the norm is $N(x + y\sqrt{2}) = x^2 - 2y^2$; the degree of the extension is 2. It is straightforward to check that the ring $\mathfrak{o}_K$ of integers of K is $\mathbb{Z}[\sqrt{2}]$. This ring is easily seen to be principal; in fact, it is even euclidian. The roots are unity are $\{1, -1\}$, and the units are generated by the fundamental unit $\sqrt{2} - 1$.

The (global) different is easily seen to be $\mathfrak{D} = (2\sqrt{2}) = (\sqrt{2})^3$, and the discriminant is 8.

We now consider a few particular places.

1. The prime 2.

Let $\mathfrak{p} = (2)$ and v be the corresponding place. The only prime of $\mathbb{Z}[\sqrt{2}]$ lying above $\mathfrak{p}$ is $\mathfrak{P} = (\sqrt{2})$. Let w be the corresponding place. We have $\mathfrak{p}\mathfrak{o}_K = \mathfrak{P}^2$: in other words, the prime 2 ramifies in $\mathbb{Z}[\sqrt{2}]$, and the ramification index is $e = 2$; the prime $\mathfrak{P}$ is totally and strongly ramified. The residue field is $\mathbb{Z}[\sqrt{2}]/\mathfrak{P} = \mathbb{F}_2$, which is equal to $\mathbb{Z}/\mathfrak{p}$: in other words, the modular degree is $f = 1$. The local degree is $ef = 2$, which means that the local field $K_w = \mathbb{Q}(\sqrt{2})_w$ is of degree 2 over $k_v = \mathbb{Q}_v = \mathbb{Q}_2$. Indeed, $\mathbb{Q}(\sqrt{2})_w = \mathbb{Q}_2(\sqrt{2})$. The local trace and norm extend the global trace and norm; in fact, $\mathrm{Tr}_w(x+y\sqrt{2}) = 2x$ and $N_w(x+y\sqrt{2}) = x^2 - 2y^2$. The decomposition group $G_\mathfrak{P}$ of the prime $\mathfrak{P}$ is $G = \{1, \sigma\}$; the galois group $\bar{G}_\mathfrak{P}$ of the residue field extension $\mathbb{F}_2/\mathbb{F}_2$ is $\{1\}$. The inertia group is therefore $G = \{1, \sigma\}$. The decomposition field is $\mathbb{Q}$, and the inertia field is also $\mathbb{Q}$. The local different (that is, the $\mathfrak{P}$-component of the different) is $\mathfrak{D}_\mathfrak{P} = \mathfrak{P}^3$, witnessing again the fact that the prime in question is strongly ramified. Writing $|\cdot|_w$ for the absolute value on K_w which extends $|\cdot|_v = |\cdot|_2$, and $\|\cdot\|_w$ for the normalized absolute value on K_w (the module), we have $|2|_w = \frac{1}{2}$, $|\sqrt{2}|_w = \frac{1}{\sqrt{2}}$, and $\|2\|_w = \frac{1}{4}$, $\|\sqrt{2}\|_w = \frac{1}{2}$; in fact, $\|\cdot\|_w = |\cdot|_w^2$ because 2 is the local degree.

2. The prime 3.

Let $\mathfrak{p} = (3)$ and v be the corresponding place. The only prime of $\mathbb{Z}[\sqrt{2}]$ lying above $\mathfrak{p}$ is $\mathfrak{P} = (3)$. Let w be the corresponding place. We have $\mathfrak{p}\mathfrak{o}_K = \mathfrak{P}$: in other words, the prime 3 does not ramify in $\mathbb{Z}[\sqrt{2}]$, that is, the ramification index is $e = 1$. The residue field is $\mathbb{Z}[\sqrt{2}]/\mathfrak{P} = \mathbb{F}_9$, which is quadratic over $\mathbb{Z}/\mathfrak{p}$: in other words, the modular degree is $f = 2$. The local degree is $ef = 2$, which means that the local field $K_w = \mathbb{Q}(\sqrt{2})_w$ is of degree 2 over $k_v = \mathbb{Q}_v = \mathbb{Q}_3$. Indeed, $\mathbb{Q}(\sqrt{2})_w = \mathbb{Q}_3(\sqrt{2})$. The local trace and norm extend the global trace and norm; in fact, $\mathrm{Tr}_w(x+y\sqrt{2}) = 2x$ and $N_w(x+y\sqrt{2}) = x^2 - 2y^2$. The decomposition group $G_\mathfrak{P}$ of the prime $\mathfrak{P}$ is $G = \{1, \sigma\}$; the galois group $\bar{G}_\mathfrak{P}$ of the residue field extension $\mathbb{F}_9/\mathbb{F}_3$ is $\{1, \bar{\sigma}\}$. The inertia group is therefore $\{1\}$. The decomposition field is $\mathbb{Q}$, and the inertia field is $\mathbb{Q}(\sqrt{2})$. The local different (that is, the $\mathfrak{P}$-component of the different) is $\mathfrak{D}_\mathfrak{P} = (1)$, witnessing again the fact that the prime in question is unramified. Writing $|\cdot|_w$ for the absolute value on K_w which extends $|\cdot|_v = |\cdot|_3$, and $\|\cdot\|_w$ for the normalized absolute value on K_w (the module), we have $|3|_w = \frac{1}{3}$, $|\sqrt{2}|_w = 1$, and $\|3\|_w = \frac{1}{9}$, $\|\sqrt{2}\|_w = 1$; in fact, $\|\cdot\|_w = |\cdot|_w^2$ because 2 is the local degree.

3. The prime 7.

Let $\mathfrak{p} = (7)$ and v be the corresponding place. There are two primes of $\mathbb{Z}[\sqrt{2}]$ lying above $\mathfrak{p}$, namely $\mathfrak{P} = (3 + \sqrt{2})$, and $\mathfrak{P}' = (3 - \sqrt{2})$. Let w, w' be the corresponding places. We have $\mathfrak{p}\mathfrak{o}_K = \mathfrak{P}\mathfrak{P}'$: in other words, the prime 7 splits in $\mathbb{Z}[\sqrt{2}]$. Since neither prime ramifies, the two ramification indices are $e = e' = 1$. The residue fields are $\mathbb{Z}[\sqrt{2}]/\mathfrak{P} = \mathbb{F}_7$ and $\mathbb{Z}[\sqrt{2}]/\mathfrak{P}' = \mathbb{F}_7$, which are both equal to $\mathbb{Z}/\mathfrak{p}$: in other words, the modular degrees are $f = f' = 1$. Note that though the two residue fields are equal, the canonical maps from $\mathbb{Z}[\sqrt{2}]$ to one and the other are *not* equal: in $\mathbb{Z}[\sqrt{2}]/\mathfrak{P}$, we have $\sqrt{2} = 4$, whereas in $\mathbb{Z}[\sqrt{2}]/\mathfrak{P}'$ we have $\sqrt{2} = 3$. The equality of the ramification indices and the modular degrees was necessary, since the extension K/k is Galois, so the Galois group operates transitively on the primes lying above $\mathfrak{p}$; in this case, $\sigma\mathfrak{P} = \mathfrak{P}'$ and $\sigma\mathfrak{P}' = \mathfrak{P}$. The local degrees are $ef = e'f' = 1$, which means that the local fields $K_w = \mathbb{Q}_w$ and $K_{w'} = \mathbb{Q}_{w'}$ are equal to $k_v = \mathbb{Q}_v = \mathbb{Q}_7$. Indeed, the polynomial $X^2 - 2$ is split over $\mathbb{Q}_7$ as $X^2 - 2 = (X - ...454)(X - ...213)$. Note that though the two local fields are equal, the canonical embeddings from $\mathbb{Z}[\sqrt{2}]$ to one and the other are *not* equal. Both the local trace and the local norm are the identity on $K_w = K_{w'} = k_v$; however, they are not equal on K (since the embedding of K in K_w is not equal to that of K in $K_{w'}$). The decomposition group $G_\mathfrak{P}$ of the prime $\mathfrak{P}$ is $G = \{1\}$; the galois group $\bar{G}_\mathfrak{P}$ of the residue field extension $\mathbb{F}_7/\mathbb{F}_7$ is $\{1\}$. The inertia group is therefore $\{1\}$. The decomposition field is $\mathbb{Q}(\sqrt{2})$, and the inertia field is also $\mathbb{Q}(\sqrt{2})$. The local different (that is, the $\mathfrak{P}$-component of the different) is $\mathfrak{D}_\mathfrak{P} = (1)$, witnessing again the fact that the prime in question is unramified. Writing $|\cdot|_w$ for the absolute value on K_w which extends $|\cdot|_v = |\cdot|_7$, and $||\cdot||_w$ for the normalized absolute value on K_w (the module), we have $|7|_w = \frac{1}{7}$, $|3+\sqrt{2}|_w = \frac{1}{7}$, $|3-\sqrt{2}|_w = 1$, $|\sqrt{2}|_w = 1$, and $||7||_w = \frac{1}{7}$, $||3+\sqrt{2}||_w = \frac{1}{7}$, $||3-\sqrt{2}||_w = 1$, $||\sqrt{2}||_w = 1$; in fact, $||\cdot||_w = |\cdot|_w$ because 1 is the local degree. (Of course, all the final discussion applies, *mutatis mutandis*, to w' and $\mathfrak{P}'$ instead of w and $\mathfrak{P}$).

4. The infinite places.

Let $v = \infty$ be the infinite place of $\mathbb{Q}$ (it is real). There are two infinite places, w, w', lying above v, both real: let w be the one of the two which is given by the canonical embedding of $\mathbb{Q}(\sqrt{2})$ into $\mathbb{R}$, whereas w' is given by the embedding σ. The local degree is 1, which means that the local fields $K_w = \mathbb{Q}(\sqrt{2})_w$ and $K_{w'} = \mathbb{Q}(\sqrt{2})_{w'}$ are both equal to $k_v = \mathbb{Q}_v = \mathbb{R}$. Indeed, the polynomial X^2-2 is split over $\mathbb{R}$ as $X^2-2 = (X-1.41...)(X+1.41...)$. Note that though the two local fields are equal, the canonical embeddings from $\mathbb{Z}[\sqrt{2}]$ to one and the other are *not* equal. Both the local trace and the local norm are the identity on $K_w = K_{w'} = k_v$; however, they are not equal on K (since the embedding of K in K_w is not equal to that of K in $K_{w'}$). Writing $|\cdot|_w$ for the absolute value on K_w which extends $|\cdot|_v = |\cdot|$, and $||\cdot||_w$ for the normalized absolute value on K_w (the module), we have $|2|_w = 2$, $|\sqrt{2}|_w = \sqrt{2}$, and $||2||_w = 2$, $||\sqrt{2}||_w = \sqrt{2}$; in fact, $||\cdot||_w = |\cdot|_w$ because 1 is the local degree. (Of course, all the final discussion applies, *mutatis mutandis*, to w' instead of w).

Witnessing the product formula, we have, e.g.:

$$||2||_{\sqrt{2}} \times ||2||_3 \times ||2||_{3+\sqrt{2}} \times ||2||_{3-\sqrt{2}} \times \cdots \times ||2||_\infty \times ||2||_{\infty'}$$
$$= \frac{1}{4} \times 1 \times 1 \times 1 \times \cdots \times 2 \times 2 = 1$$

$$||\sqrt{2}||_{\sqrt{2}} \times ||\sqrt{2}||_3 \times ||\sqrt{2}||_{3+\sqrt{2}} \times ||\sqrt{2}||_{3-\sqrt{2}} \times \cdots \times ||\sqrt{2}||_\infty \times ||\sqrt{2}||_{\infty'}$$
$$= \frac{1}{2} \times 1 \times 1 \times 1 \times \cdots \times \sqrt{2} \times \sqrt{2} = 1$$

$$||3||_{\sqrt{2}} \times ||3||_3 \times ||3||_{3+\sqrt{2}} \times ||3||_{3-\sqrt{2}} \times \cdots \times ||3||_\infty \times ||3||_{\infty'}$$
$$= 1 \times \frac{1}{9} \times 1 \times 1 \times \cdots \times 3 \times 3 = 1$$

$$||7||_{\sqrt{2}} \times ||7||_3 \times ||7||_{3+\sqrt{2}} \times ||7||_{3-\sqrt{2}} \times \cdots \times ||7||_\infty \times ||7||_{\infty'}$$
$$= 1 \times 1 \times \frac{1}{7} \times \frac{1}{7} \times \cdots \times 7 \times 7 = 1$$

$$||3+\sqrt{2}||_{\sqrt{2}} \times ||3+\sqrt{2}||_3 \times ||3+\sqrt{2}||_{3+\sqrt{2}} \times ||3+\sqrt{2}||_{3-\sqrt{2}} \times \cdots \times ||3+\sqrt{2}||_\infty \times ||3+\sqrt{2}||_{\infty'}$$
$$= 1 \times 1 \times \frac{1}{7} \times 1 \times \cdots \times (3+\sqrt{2}) \times (3-\sqrt{2}) = 1$$

Cube root of unity.

We consider the field extension $K/k := \mathbb{Q}(\omega)/\mathbb{Q}$, where $\omega = e^{2i\pi/3}$. This extension is Galois with Galois group $\{1, \sigma\}$, satisfying $\sigma^2 = 1$, where σ is complex conjugation, $\sigma(\omega) = \omega^2 = -1-\omega$. A basis of K over k is given by $1, \omega$; the trace is $\mathrm{Tr}(x + y\omega) = 2x - y$ and the norm is $N(x + y\omega) = x^2 - xy + y^2$; the degree of the extension is 2. It is straightforward to check that the ring $\mathfrak{o}_K$ of integers of K is $\mathbb{Z}[\omega]$. This ring is easily seen to be principal; in fact, it is even euclidian. The roots are unity are $\{1, \omega, \omega^2\}$, and the units are $\{1, \omega, \omega^2\}$.

The (global) different is easily seen to be $\mathfrak{D} = (1 - \omega) = (1 + 2\omega)$, and the discriminant is -3.

We now consider a few particular places.

1. The prime 2.

Let $\mathfrak{p} = (2)$ and v be the corresponding place. The only prime of $\mathbb{Z}[\omega]$ lying above $\mathfrak{p}$ is $\mathfrak{P} = (2)$. Let w be the corresponding place. We have $\mathfrak{p}\mathfrak{o}_K = \mathfrak{P}$: in other words, the prime 2 does not ramify in $\mathbb{Z}[\omega]$, that is, the ramification index is $e = 1$. The residue field is $\mathbb{Z}[\omega]/\mathfrak{P} = \mathbb{F}_4$, which is quadratic over $\mathbb{Z}/\mathfrak{p}$: in other words, the modular degree is $f = 2$. The local degree is $ef = 2$, which means that the local field $K_w = \mathbb{Q}(\omega)_w$ is of degree 2 over $k_v = \mathbb{Q}_v = \mathbb{Q}_2$. Indeed, $\mathbb{Q}(\omega)_w = \mathbb{Q}_2(\omega)$ (where ω is a root over $\mathbb{Q}_2$ of the irreducible polynomial $X^2 + X + 1$). The local trace and norm extend the global trace and norm; in fact, $\mathrm{Tr}_w(x + y\omega) = 2x - y$ and $N_w(x + y\omega) = x^2 - xy + y^2$. The decomposition group $G_\mathfrak{P}$ of the prime $\mathfrak{P}$ is $G = \{1, \sigma\}$; the galois group $\bar{G}_\mathfrak{P}$ of the residue field extension $\mathbb{F}_4/\mathbb{F}_2$ is $\{1, \bar{\sigma}\}$. The inertia group is therefore $\{1\}$. The decomposition field is $\mathbb{Q}$, and the inertia field is $\mathbb{Q}(\sqrt{2})$. The local different (that is, the $\mathfrak{P}$-component of the different) is $\mathfrak{D}_\mathfrak{P} = (1)$, witnessing again the fact that the prime in question is unramified. Writing $|\cdot|_w$ for the absolute value on K_w which extends $|\cdot|_v = |\cdot|_2$, and $\|\cdot\|_w$ for the normalized absolute value on K_w (the module), we have $|2|_w = \frac{1}{2}$, $|\omega|_w = 1$, and $\|2\|_w = \frac{1}{4}$, $\|\omega\|_w = 1$; in fact, $\|\cdot\|_w = |\cdot|_w^2$ because 2 is the local degree.

2. The prime 3.

Let $\mathfrak{p} = (3)$ and v be the corresponding place. The only prime of $\mathbb{Z}[\omega]$ lying above $\mathfrak{p}$ is $\mathfrak{P} = (1 - \omega) = (1 + 2\omega)$. Let w be the corresponding place. We have $\mathfrak{p}\mathfrak{o}_K = \mathfrak{P}^2$: in other words, the prime 3 ramifies in $\mathbb{Z}[\omega]$, and the ramification index is $e = 2$; the prime $\mathfrak{P}$ is totally and tamely ramified. The residue field is $\mathbb{Z}[\omega]/\mathfrak{P} = \mathbb{F}_3$, which is equal to $\mathbb{Z}/\mathfrak{p}$: in other words, the modular degree is $f = 1$. The local degree is $ef = 2$, which means that the local field $K_w = \mathbb{Q}(\omega)_w$ is of degree 2 over $k_v = \mathbb{Q}_v = \mathbb{Q}_3$. Indeed, $\mathbb{Q}(\omega)_w = \mathbb{Q}_3(\omega)$ (where ω is a root over $\mathbb{Q}_3$ of the irreducible polynomial $X^2 + X + 1$). The local trace and norm extend the global trace and norm; in fact, $\mathrm{Tr}_w(x + y\omega) = 2x - y$ and $N_w(x + y\omega) = x^2 - xy + y^2$. The decomposition group $G_\mathfrak{P}$ of the prime $\mathfrak{P}$ is $G = \{1, \sigma\}$; the galois group $\bar{G}_\mathfrak{P}$ of the residue field extension $\mathbb{F}_3/\mathbb{F}_3$ is $\{1\}$. The inertia group is therefore $G = \{1, \sigma\}$. The decomposition field is $\mathbb{Q}$, and the inertia field is also $\mathbb{Q}$. The local different (that is, the $\mathfrak{P}$-component of the different) is $\mathfrak{D}_\mathfrak{P} = \mathfrak{P}$, witnessing again the fact that the prime in question is tamely ramified. Writing $|\cdot|_w$ for the absolute value on K_w which extends $|\cdot|_v = |\cdot|_3$, and $\|\cdot\|_w$ for the normalized absolute value on K_w (the module), we have $|3|_w = \frac{1}{3}$, $|1 - \omega|_w = \frac{1}{\sqrt{3}}$, $|\omega|_w = 1$, and $\|3\|_w = \frac{1}{9}$, $\|1 - \omega\|_w = \frac{1}{3}$, $\|\omega\|_w = 1$; in fact, $\|\cdot\|_w = |\cdot|_w^2$ because 2 is the local degree.

3. The prime 7.

Let $\mathfrak{p} = (7)$ and v be the corresponding place. There are two primes of $\mathbb{Z}[\omega]$ lying above $\mathfrak{p}$, namely $\mathfrak{P} = (2-\omega)$, and $\mathfrak{P}' = (3+\omega)$. Let w, w' be the corresponding places. We have $\mathfrak{p}\mathfrak{o}_K = \mathfrak{P}\mathfrak{P}'$: in other words, the prime 7 splits in $\mathbb{Z}[\omega]$. Since neither prime ramifies, the two ramification indices are $e = e' = 1$. The residue fields are $\mathbb{Z}[\omega]/\mathfrak{P} = \mathbb{F}_7$ and $\mathbb{Z}[\omega]/\mathfrak{P}' = \mathbb{F}_7$, which are both equal to $\mathbb{Z}/\mathfrak{p}$: in other words, the modular degrees are $f = f' = 1$. Note that though the two residue fields are equal, the canonical maps from $\mathbb{Z}[\omega]$ to one and the other are *not* equal: in $\mathbb{Z}[\omega]/\mathfrak{P}$, we have $\omega = 2$, whereas in $\mathbb{Z}[\omega]/\mathfrak{P}'$ we have $\omega = 4$. The equality of the ramification indices and the modular degrees was necessary, since the extension K/k is Galois, so the Galois group operates transitively on the primes lying above $\mathfrak{p}$; in this case, $\sigma\mathfrak{P} = \mathfrak{P}'$ and $\sigma\mathfrak{P}' = \mathfrak{P}$. The local degrees are $ef = e'f' = 1$, which means that the local fields $K_w = \mathbb{Q}_w$ and $K_{w'} = \mathbb{Q}_{w'}$ are equal to $k_v = \mathbb{Q}_v = \mathbb{Q}_7$. Indeed, the polynomial $X^2 + X + 1$ is split over $\mathbb{Q}_7$ as $X^2 + X + 1 = (X - ..642)(X - ...024)$. Note that though the two local fields are equal, the canonical embeddings from $\mathbb{Z}[\omega]$ to one and the other are *not* equal. Both the local trace and the local norm are the identity on $K_w = K_{w'} = k_v$; however, they are not equal on K (since the embedding of K in K_w is not equal to that of K in $K_{w'}$). The decomposition group $G_\mathfrak{P}$ of the prime $\mathfrak{P}$ is $G = \{1\}$; the galois group $\bar{G}_\mathfrak{P}$ of the residue field extension $\mathbb{F}_7/\mathbb{F}_7$ is $\{1\}$. The inertia group is therefore $\{1\}$. The decomposition field is $\mathbb{Q}(\omega)$, and the inertia field is also $\mathbb{Q}(\omega)$. The local different (that is, the $\mathfrak{P}$-component of the different) is $\mathfrak{D}_{\mathfrak{P}} = (1)$, witnessing again the fact that the prime in question is unramified. Writing $|\cdot|_w$ for the absolute value on K_w which extends $|\cdot|_v = |\cdot|_7$, and $\|\cdot\|_w$ for the normalized absolute value on K_w (the module), we have $|7|_w = \frac{1}{7}$, $|2-\omega|_w = \frac{1}{7}$, $|3+\omega|_w = 1$, $|\omega|_w = 1$, and $\|7\|_w = \frac{1}{7}$, $\|2-\omega\|_w = \frac{1}{7}$, $\|3+\omega\|_w = 1$, $\|\omega\|_w = 1$; in fact, $\|\cdot\|_w = |\cdot|_w$ because 1 is the local degree. (Of course, all the final discussion applies, *mutatis mutandis*, to w' and $\mathfrak{P}'$ instead of w and $\mathfrak{P}$).

4. The infinite place.

Let $v = \infty$ be the infinite place of $\mathbb{Q}$ (it is real). The only place of $\mathbb{Q}(\omega)$ lying above v is a complex place w. The local degree is 2, which means that the local field $K_w = \mathbb{Q}(\omega)_w$ is of degree 2 over $k_v = \mathbb{Q}_v = \mathbb{R}$. Indeed, $\mathbb{Q}(i)_w = \mathbb{C}$. The local trace and norm extend the global trace and norm; in fact, $\mathrm{Tr}_w(x+y\omega) = 2x-y$ and $N_w(x+y\omega) = x^2 - xy + y^2$. Writing $|\cdot|_w$ for the absolute value on K_w which extends $|\cdot|_v = |\cdot|$, and $\|\cdot\|_w$ for the normalized absolute value on K_w (the module), we have $|2|_w = 2$, $|\omega|_w = 1$, and $\|2\|_w = 4$, $\|\omega\|_w = 1$; in fact, $\|\cdot\|_w = |\cdot|_w^2$ because 2 is the local degree.

Witnessing the product formula, we have, e.g.:

$$\begin{aligned}&\|\omega\|_2 \times \|\omega\|_{1-\omega} \times \|\omega\|_{2-\omega} \times \|\omega\|_{3+\omega} \times \cdots \times \|\omega\|_\infty\\ =&1 \times 1 \times 1 \times 1 \times \cdots \times 1 = 1\end{aligned}$$

$$\begin{aligned}&\|2\|_2 \times \|2\|_{1-\omega} \times \|2\|_{2-\omega} \times \|2\|_{3+\omega} \times \cdots \times \|2\|_\infty\\ =&\frac{1}{4} \times 1 \times 1 \times 1 \times \cdots \times 4 = 1\end{aligned}$$

$$\begin{aligned}&\|3\|_2 \times \|3\|_{1-\omega} \times \|3\|_{2-\omega} \times \|3\|_{3+\omega} \times \cdots \times \|3\|_\infty\\ =&1 \times \frac{1}{9} \times 1 \times 1 \times \cdots \times 9 = 1\end{aligned}$$

$$\begin{aligned}&\|1-\omega\|_2 \times \|1-\omega\|_{1-\omega} \times \|1-\omega\|_{2-\omega} \times \|1-\omega\|_{3+\omega} \times \cdots \times \|1-\omega\|_\infty\\ =&1 \times \frac{1}{3} \times 1 \times 1 \times \cdots \times 3 = 1\end{aligned}$$

$$\begin{aligned}&\|7\|_2 \times \|7\|_{1-\omega} \times \|7\|_{2-\omega} \times \|7\|_{3+\omega} \times \cdots \times \|7\|_\infty\\ =&1 \times 1 \times \frac{1}{7} \times \frac{1}{7} \times \cdots \times 49 = 1\end{aligned}$$

$$\begin{aligned}&\|2-\omega\|_2 \times \|2-\omega\|_{1-\omega} \times \|2-\omega\|_{2-\omega} \times \|2-\omega\|_{3+\omega} \times \cdots \times \|2-\omega\|_\infty\\ =&1 \times 1 \times \frac{1}{7} \times 1 \times \cdots \times 7 = 1\end{aligned}$$

Cube root of 2.

We consider the field extension $K/k := \mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$. This extension is not Galois and the Galois group $\{1\}$. A basis of K over k is given by $1, \sqrt[3]{2}, \sqrt[3]{4}$; the trace is $\mathrm{Tr}(x + y\sqrt[3]{2} + z\sqrt[3]{4}) = 3x$ and the norm is $N(x + y\sqrt[3]{2} + z\sqrt[3]{4}) = x^3 + 2y^3 + 4z^3 - 6xyz$; the degree of the extension is 3. It is straightforward (but painful) to check that the ring $\mathfrak{o}_K$ of integers of K is $\mathbb{Z}[\sqrt[3]{2}]$. The roots are unity are $\{1, -1\}$, and the units are generated by the fundamental unit $\sqrt[3]{2} - 1$.

The (global) different is easily seen to be $\mathfrak{D} = (3\sqrt[3]{4})$, and the discriminant is $-108 = -2^2 \times 3^3$.

We now consider a few particular places.

1. The prime 2.

   Let $\mathfrak{p} = (2)$ and v be the corresponding place. The only prime of $\mathbb{Z}[\sqrt[3]{2}]$ lying above $\mathfrak{p}$ is $\mathfrak{P} = (\sqrt[3]{2})$. Let w be the corresponding place. We have $\mathfrak{p}\mathfrak{o}_K = \mathfrak{P}^3$: in other words, the prime 2 ramifies in $\mathbb{Z}[\sqrt[3]{2}]$, and the ramification index is $e = 3$; the prime $\mathfrak{P}$ is totally and tamely ramified. The residue field is $\mathbb{Z}[\sqrt[3]{2}]/\mathfrak{P} = \mathbb{F}_2$, which is equal to $\mathbb{Z}/\mathfrak{p}$: in other words, the modular degree is $f = 1$. The local degree is $ef = 3$, which means that the local field $K_w = \mathbb{Q}(\sqrt[3]{2})_w$ is of degree 3 over $k_v = \mathbb{Q}_v = \mathbb{Q}_2$. Indeed, $\mathbb{Q}(\sqrt[3]{2})_w = \mathbb{Q}_2(\sqrt[3]{2})$. The local trace and norm extend the global trace and norm; in fact, $\mathrm{Tr}_w(x+y\sqrt[3]{2}+z\sqrt[3]{4}) = 3x$ and $N_w(x+y\sqrt[3]{2}+z\sqrt[3]{4}) = x^3+2y^3+4z^3-6xyz$. The local different (that is, the $\mathfrak{P}$-component of the different) is $\mathfrak{D}_{\mathfrak{P}} = \mathfrak{P}^2$, witnessing again the fact that the prime in question is tamely ramified. Writing $|\cdot|_w$ for the absolute value on K_w which extends $|\cdot|_v = |\cdot|_2$, and $\|\cdot\|_w$ for the normalized absolute value on K_w (the module), we have $|2|_w = \frac{1}{2}$, $|\sqrt[3]{2}|_w = \frac{1}{\sqrt[3]{2}}$, and $\|2\|_w = \frac{1}{8}$, $\|\sqrt[3]{2}\|_w = \frac{1}{2}$; in fact, $\|\cdot\|_w = |\cdot|_w^3$ because 3 is the local degree.

2. The prime 3.

   Let $\mathfrak{p} = (3)$ and v be the corresponding place. The only prime of $\mathbb{Z}[\sqrt[3]{2}]$ lying above $\mathfrak{p}$ is $\mathfrak{P} = (1 + \sqrt[3]{2})$. Let w be the corresponding place. We have $\mathfrak{p}\mathfrak{o}_K = \mathfrak{P}^3$: in other words, the prime 3 ramifies in $\mathbb{Z}[\sqrt[3]{2}]$, and the ramification index is $e = 3$; the prime $\mathfrak{P}$ is totally and strongly ramified. The residue field is $\mathbb{Z}[\sqrt[3]{2}]/\mathfrak{P} = \mathbb{F}_2$, which is equal to $\mathbb{Z}/\mathfrak{p}$: in other words, the modular degree is $f = 1$. The local degree is $ef = 3$, which means that the local field $K_w = \mathbb{Q}(\sqrt[3]{2})_w$ is of degree 3 over $k_v = \mathbb{Q}_v = \mathbb{Q}_3$. Indeed, $\mathbb{Q}(\sqrt[3]{2})_w = \mathbb{Q}_2(\sqrt[3]{2})$. The local trace and norm extend the global trace and norm; in fact, $\mathrm{Tr}_w(x+y\sqrt[3]{2}+z\sqrt[3]{4}) = 3x$ and $N_w(x+y\sqrt[3]{2}+z\sqrt[3]{4}) = x^3+2y^3+4z^3-6xyz$. The local different (that is, the $\mathfrak{P}$-component of the different) is $\mathfrak{D}_{\mathfrak{P}} = \mathfrak{P}^3$, witnessing again the fact that the prime in question is strongly ramified. Writing $|\cdot|_w$ for the absolute value on K_w which extends $|\cdot|_v = |\cdot|_2$, and $\|\cdot\|_w$ for the normalized absolute value on K_w (the module), we have $|3|_w = \frac{1}{3}$, $|1 + \sqrt[3]{2}|_w = \frac{1}{\sqrt[3]{3}}$, and $\|2\|_w = \frac{1}{8}$, $\|1 + \sqrt[3]{2}\|_w = \frac{1}{3}$; in fact, $\|\cdot\|_w = |\cdot|_w^3$ because 3 is the local degree.

3. The prime 5.

   Let $\mathfrak{p} = (7)$ and v be the corresponding place. There are two primes of $\mathbb{Z}[\sqrt[3]{2}]$ lying above $\mathfrak{p}$, namely $\mathfrak{P} = (1 + \sqrt[3]{4})$, and $\mathfrak{P}' = (1 + 2\sqrt[3]{2} - \sqrt[3]{4})$. Let w, w' be the corresponding places. We have $\mathfrak{p}\mathfrak{o}_K = \mathfrak{P}\mathfrak{P}'$: in other words, the prime 5 splits, but not completely, in $\mathbb{Z}[\sqrt[3]{2}]$. Since neither prime ramifies, the two ramification indices are $e = e' = 1$. The residue fields are $\mathbb{Z}[\sqrt[3]{2}]/\mathfrak{P} = \mathbb{F}_5$ and $\mathbb{Z}[\sqrt[3]{2}]/\mathfrak{P}' = \mathbb{F}_{25}$, which respectively equal to, and quadratic over, $\mathbb{Z}/\mathfrak{p}$: in other words, the modular degrees are $f = 1$ and $f' = 2$. The local degrees are $ef = 1$ and $e'f' = 2$, which means that the local fields $K_w = \mathbb{Q}_w$ and $K_{w'} = \mathbb{Q}_{w'}$ are respecively of degree 1 and 2 over $k_v = \mathbb{Q}_v = \mathbb{Q}_5$. Indeed, the polynomial $X^3 - 2$ is decomposes over $\mathbb{Q}_5$ as $X^3 - 2 = (X - \ldots 203)(X^2 + \ldots 203X + \ldots 214)$; in fact, $K_{w'} = \mathbb{Q}_5(\sqrt{\ldots 243})$ (because $\ldots 243$ is the discriminant of the polynomial $X^2 + \ldots 203X + \ldots 214$). Both the local trace and the local norm are the identity on $K_w = k_v$; on $K_{w'}$, the trace is $\mathrm{Tr}(x + y\sqrt{\ldots 243}) = 2x$ and the norm is $N(x + y\sqrt{\ldots 243}) = x^2 - \ldots 243y^2$. The local differents (that is, the $\mathfrak{P}$- and $\mathfrak{P}'$-components of the different) are $\mathfrak{D}_{\mathfrak{P}} = \mathfrak{D}_{\mathfrak{P}'} = (1)$, witnessing again the fact that the primes in question are unramified. Writing $|\cdot|_w$ for the absolute value on K_w which extends $|\cdot|_v = |\cdot|_5$, and $\|\cdot\|_w$ for the normalized absolute value on K_w (the module), and $|\cdot|_{w'}$ and $\|\cdot\|_{w'}$ for the corresponding objects for $K_{w'}$, we have $|5|_w = \frac{1}{5}$, $|1 + \sqrt[3]{4}|_w = \frac{1}{5}$, $|1 + 2\sqrt[3]{2} - \sqrt[3]{4}|_w = 1$, and $\|5\|_w = \frac{1}{5}$, $\|1 + \sqrt[3]{4}\|_w = \frac{1}{5}$, $\|1 + 2\sqrt[3]{2} - \sqrt[3]{4}\|_w = 1$, in fact, $\|\cdot\|_w = |\cdot|_w$ because 1 is the local degree; on the other hand, we have $|5|_{w'} = \frac{1}{5}$, $|1 + \sqrt[3]{4}|_{w'} = 1$, $|1 + 2\sqrt[3]{2} - \sqrt[3]{4}|_{w'} = \frac{1}{5}$, and $\|5\|_{w'} = \frac{1}{25}$, $\|1 + \sqrt[3]{4}\|_{w'} = 1$, $\|1 + 2\sqrt[3]{2} - \sqrt[3]{4}\|_{w'} = \frac{1}{25}$, in fact, $\|\cdot\|_{w'} = |\cdot|_{w'}^2$ because 2 is the local degree.

4. The prime 7.

   Let $\mathfrak{p} = (7)$ and v be the corresponding place. The only prime of $\mathbb{Z}[\sqrt[3]{2}]$ lying above $\mathfrak{p}$ is $\mathfrak{P} = (7)$. Let w be the corresponding place. We have $\mathfrak{p}\mathfrak{o}_K = \mathfrak{P}$: in other words, the prime 7 does not ramify in $\mathbb{Z}[\sqrt[3]{2}]$, that is, the ramification index is $e = 1$. The residue field is $\mathbb{Z}[\sqrt[3]{2}]/\mathfrak{P} = \mathbb{F}_{343}$, which is cubic over $\mathbb{Z}/\mathfrak{p}$: in other words, the modular degree is $f = 3$. The local degree is $ef = 3$, which means that the local field $K_w = \mathbb{Q}(\sqrt[3]{2})_w$ is of degree 3 over $k_v = \mathbb{Q}_v = \mathbb{Q}_7$. Indeed, $\mathbb{Q}(\sqrt[3]{2})_w = \mathbb{Q}_7(\sqrt[3]{2})$. The local trace and norm extend the global trace and norm; in fact, $\mathrm{Tr}_w(x + y\sqrt[3]{2} + z\sqrt[3]{4}) = 3x$ and $N_w(x + y\sqrt[3]{2} + z\sqrt[3]{4}) = x^3 + 2y^2 + 4z^2 - 6xyz$. The local different (that is, the $\mathfrak{P}$-component of the different) is $\mathfrak{D}_{\mathfrak{P}} = (1)$, witnessing again the fact that the prime in question is unramified. Writing $|\cdot|_w$ for the absolute value on K_w which extends $|\cdot|_v = |\cdot|_7$, and $\|\cdot\|_w$ for the normalized absolute value on K_w (the module), we have $|7|_w = \frac{1}{7}$, $|\sqrt[3]{2}|_w = 1$, and $\|7\|_w = \frac{1}{343}$, $\|\sqrt[3]{2}\|_w = 1$; in fact, $\|\cdot\|_w = |\cdot|_w^3$ because 3 is the local degree.

5. The prime 31.

Let $\mathfrak{p} = (31)$ and v be the corresponding place. There are three primes of $\mathbb{Z}[\sqrt[3]{2}]$ lying above $\mathfrak{p}$, namely $\mathfrak{P} = (3+\sqrt[3]{4})$, $\mathfrak{P}' = (1-2\sqrt[3]{2}-\sqrt[3]{4})$, and $\mathfrak{P}'' = (1-2\sqrt[3]{4})$. Let w, w', w'' be the corresponding places. We have $\mathfrak{p}\mathfrak{o}_K = \mathfrak{P}\mathfrak{P}'\mathfrak{P}''$: in other words, the prime 31 splits completely in $\mathbb{Z}[\sqrt[3]{2}]$. Since none of the primes ramifies, the ramification indices are $e = e' = e'' = 1$. The residue fields are $\mathbb{Z}[\sqrt[3]{2}]/\mathfrak{P} = \mathbb{F}_{31}$, $\mathbb{Z}[\sqrt[3]{2}]/\mathfrak{P}' = \mathbb{F}_{31}$, and $\mathbb{Z}[\sqrt[3]{2}]/\mathfrak{P}'' = \mathbb{F}_{31}$, which are all equal to $\mathbb{Z}/\mathfrak{p}$: in other words, the modular degrees are $f = f' = f'' = 1$. Note that though the two residue fields are equal, the canonical maps from $\mathbb{Z}[\sqrt[3]{2}]$ to one and the other are *not* equal: in $\mathbb{Z}[\sqrt[3]{2}]/\mathfrak{P}$, we have $\sqrt[3]{2} = 20$, whereas in $\mathbb{Z}[\sqrt[3]{2}]/\mathfrak{P}'$ we have $\sqrt[3]{2} = 7$, and in $\mathbb{Z}[\sqrt[3]{2}]/\mathfrak{P}''$ we have $\sqrt[3]{2} = 4$. The local degrees are $ef = e'f' = e''f'' = 1$, which means that the local fields $K_w = \mathbb{Q}_w$, $K_{w'} = \mathbb{Q}_{w'}$ and $K_{w''} = \mathbb{Q}_{w''}$ are equal to $k_v = \mathbb{Q}_v = \mathbb{Q}_{31}$. Indeed, the polynomial $X^2 - 2$ is split over $\mathbb{Q}_{31}$ as $X^2 - 2 = (X - ...8:20)(X - ...13:7)(X - ...9:4)$. Note that though the three local fields are equal, the canonical embeddings from $\mathbb{Z}[\sqrt[3]{2}]$ to one and the others are *not* equal. Both the local trace and the local norm are the identity on $K_w = K_{w'} = K_{w''} = k_v$; however, they are not equal on K (since the embedding of K in K_w is not equal to that of K in $K_{w'}$). The local different (that is, the $\mathfrak{P}$-component of the different) is $\mathfrak{D}_{\mathfrak{P}} = (1)$, witnessing again the fact that the prime in question is unramified. Writing $|\cdot|_w$ for the absolute value on K_w which extends $|\cdot|_v = |\cdot|_{31}$, and $\|\cdot\|_w$ for the normalized absolute value on K_w (the module), we have $|31|_w = \frac{1}{31}$, $|3 + \sqrt[3]{4}|_w = \frac{1}{31}$, $|1 - 2\sqrt[3]{2} - \sqrt[3]{4}|_w = 1$, $|1 - 2\sqrt[3]{4}|_w = 1$, and $\|31\|_w = \frac{1}{31}$, $\|3 + \sqrt[3]{4}\|_w = \frac{1}{31}$, $\|1 - 2\sqrt[3]{2} - \sqrt[3]{4}\|_w = 1$, $\|1 - 2\sqrt[3]{4}\|_w = 1$; in fact, $\|\cdot\|_w = |\cdot|_w$ because 1 is the local degree. (Of course, all the final discussion applies, *mutatis mutandis*, to w' and $\mathfrak{P}'$ or w'' and $\mathfrak{P}''$ instead of w and $\mathfrak{P}$).

6. The infinite places.

Let $v = \infty$ be the infinite place of $\mathbb{Q}$ (it is real). There are two infinite places, w, w', lying above v, one real and one complex: let w be the real one, which is given by the canonical embedding of $\mathbb{Q}(\sqrt{2})$ into $\mathbb{R}$, and let w' be that which is given by the embedding $\sqrt[3]{2} \mapsto e^{2i\pi/3}\sqrt[3]{2}$. The local degrees of w, w' are 1 and 2 respectively, which means that the local fields $K_w = \mathbb{Q}(\sqrt[3]{2})_w$ and $K_{w'} = \mathbb{Q}(\sqrt[3]{2})_{w'}$ are respectively equal to and quadratic over $k_v = \mathbb{Q}_v = \mathbb{R}$; in fact, $K_{w'} = \mathbb{C}$. Indeed, the polynomial $X^2 - 2$ decomposes over $\mathbb{R}$ as $X^2 - 2 = (X - 1.25...)(X + 1.25... + 1.58...)$. Both the local trace and the local norm are the identity on $K_w = k_v$; on $K_{w'}$, they are the ordinary trace and norm of $\mathbb{C}$ over $\mathbb{R}$. Writing $|\cdot|_w$ for the absolute value on K_w which extends $|\cdot|_v = |\cdot|$, and $\|\cdot\|_w$ for the normalized absolute value on K_w (the module), and $|\cdot|_{w'}$ and $\|\cdot\|_{w'}$ for the corresponding objects for $K_{w'}$, we have $|2|_w = 2$, $|\sqrt[3]{2}|_w = \sqrt[3]{2}$, and $\|2\|_w = 2$, $\|\sqrt{2}\|_w = \sqrt{2}$; in fact, $\|\cdot\|_w = |\cdot|_w$ because 1 is the local degree; on the other hand, we have $|2|_{w'} = 2$, $|\sqrt[3]{2}|_{w'} = \sqrt[3]{2}$, and $\|2\|_{w'} = 4$, $\|\sqrt[3]{2}\|_{w'} = \sqrt[3]{4}$; in fact, $\|\cdot\|_w = |\cdot|_w^2$ because 2 is the local degree.

Witnessing the product formula, we have, e.g.:

$$
\begin{aligned}
&\|\sqrt[3]{2}\|_{\sqrt[3]{2}} \times \|\sqrt[3]{2}\|_{1+\sqrt[3]{2}} \times \|\sqrt[3]{2}\|_{1+\sqrt[3]{4}} \times \|\sqrt[3]{2}\|_{1+2\sqrt[3]{2}-\sqrt[3]{4}} \times \\
&\quad \|\sqrt[3]{2}\|_7 \times \|\sqrt[3]{2}\|_{3+\sqrt[3]{4}} \times \|\sqrt[3]{2}\|_{1-2\sqrt[3]{2}-\sqrt[3]{4}} \times \|\sqrt[3]{2}\|_{1-2\sqrt[3]{4}} \times \cdots \\
&\quad \times \|\sqrt[3]{2}\|_\infty \times \|\sqrt[3]{2}\|_{\infty'} \\
=&\frac{1}{2} \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times \cdots \times \sqrt[3]{2} \times \sqrt[3]{4} = 1
\end{aligned}
$$

$$
\begin{aligned}
&\|2\|_2 \times \|2\|_{1+2} \times \|2\|_{1+\sqrt[3]{4}} \times \|2\|_{1+22-\sqrt[3]{4}} \times \\
&\quad \|2\|_7 \times \|2\|_{3+\sqrt[3]{4}} \times \|2\|_{1-22-\sqrt[3]{4}} \times \|2\|_{1-2\sqrt[3]{4}} \times \cdots \\
&\quad \times \|2\|_\infty \times \|2\|_{\infty'} \\
=&\frac{1}{8} \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times \cdots \times 2 \times 4 = 1
\end{aligned}
$$

$$\begin{aligned}&||1+\sqrt[3]{2}||_{\sqrt[3]{2}}\times||1+\sqrt[3]{2}||_{1+\sqrt[3]{2}}\times||1+\sqrt[3]{2}||_{1+\sqrt[3]{4}}\times||1+\sqrt[3]{2}||_{1+2\sqrt[3]{2}-\sqrt[3]{4}}\times\\&\quad||1+\sqrt[3]{2}||_{7}\times||1+\sqrt[3]{2}||_{3+\sqrt[3]{4}}\times||1+\sqrt[3]{2}||_{1-2\sqrt[3]{2}-\sqrt[3]{4}}\times||1+\sqrt[3]{2}||_{1-2\sqrt[3]{4}}\times\cdots\\&\quad\times||1+\sqrt[3]{2}||_{\infty}\times||1+\sqrt[3]{2}||_{\infty'}\\&=1\times\frac{1}{3}\times1\times1\times1\times1\times1\times1\times\cdots\times(1+\sqrt[3]{2})\times(1-\sqrt[3]{2}+\sqrt[3]{4})=1\end{aligned}$$

$$\begin{aligned}&||3||_{\sqrt[3]{2}}\times||3||_{1+\sqrt[3]{2}}\times||3||_{1+\sqrt[3]{4}}\times||3||_{1+2\sqrt[3]{2}-\sqrt[3]{4}}\times\\&\quad||3||_{7}\times||3||_{3+\sqrt[3]{4}}\times||3||_{1-2\sqrt[3]{2}-\sqrt[3]{4}}\times||3||_{1-2\sqrt[3]{4}}\times\cdots\\&\quad\times||3||_{\infty}\times||3||_{\infty'}\\&=1\times\frac{1}{27}\times1\times1\times1\times1\times1\times1\times\cdots\times3\times9=1\end{aligned}$$

$$\begin{aligned}&||1+\sqrt[3]{4}||_{\sqrt[3]{2}}\times||1+\sqrt[3]{4}||_{1+\sqrt[3]{2}}\times||1+\sqrt[3]{4}||_{1+\sqrt[3]{4}}\times||1+\sqrt[3]{4}||_{1+2\sqrt[3]{2}-\sqrt[3]{4}}\times\\&\quad||1+\sqrt[3]{4}||_{7}\times||1+\sqrt[3]{4}||_{3+\sqrt[3]{4}}\times||1+\sqrt[3]{4}||_{1-2\sqrt[3]{2}-\sqrt[3]{4}}\times||1+\sqrt[3]{4}||_{1-2\sqrt[3]{4}}\times\cdots\\&\quad\times||1+\sqrt[3]{4}||_{\infty}\times||1+\sqrt[3]{4}||_{\infty'}\\&=1\times1\times\frac{1}{5}\times1\times1\times1\times1\times1\times\cdots\times(1+\sqrt[3]{4})\times(1+2\sqrt[3]{4}-\sqrt[3]{4})=1\end{aligned}$$

$$\begin{aligned}&||1+2\sqrt[3]{2}-\sqrt[3]{4}||_{\sqrt[3]{2}}\times||1+2\sqrt[3]{2}-\sqrt[3]{4}||_{1+\sqrt[3]{2}}\times||1+2\sqrt[3]{2}-\sqrt[3]{4}||_{1+\sqrt[3]{4}}\times||1+2\sqrt[3]{2}-\sqrt[3]{4}||_{1+2\sqrt[3]{2}-\sqrt[3]{4}}\times\\&\quad||1+2\sqrt[3]{2}-\sqrt[3]{4}||_{7}\times||1+2\sqrt[3]{2}-\sqrt[3]{4}||_{3+\sqrt[3]{4}}\times||1+2\sqrt[3]{2}-\sqrt[3]{4}||_{1-2\sqrt[3]{2}-\sqrt[3]{4}}\times||1+2\sqrt[3]{2}-\sqrt[3]{4}||_{1-2\sqrt[3]{4}}\times\cdots\\&\quad\times||1+2\sqrt[3]{2}-\sqrt[3]{4}||_{\infty}\times||1+2\sqrt[3]{2}-\sqrt[3]{4}||_{\infty'}\\&=1\times1\times1\times\frac{1}{25}\times1\times1\times1\times1\times\cdots\times(1+2\sqrt[3]{4}-\sqrt[3]{4})\times(5+5\sqrt[3]{4})=1\end{aligned}$$

$$\begin{aligned}&||7||_{\sqrt[3]{2}}\times||7||_{1+\sqrt[3]{2}}\times||7||_{1+\sqrt[3]{4}}\times||7||_{1+2\sqrt[3]{2}-\sqrt[3]{4}}\times\\&\quad||7||_{7}\times||7||_{3+\sqrt[3]{4}}\times||7||_{1-2\sqrt[3]{2}-\sqrt[3]{4}}\times||7||_{1-2\sqrt[3]{4}}\times\cdots\\&\quad\times||7||_{\infty}\times||7||_{\infty'}\\&=1\times1\times1\times1\times\frac{1}{343}\times1\times1\times1\times\cdots\times7\times49=1\end{aligned}$$

$$\begin{aligned}&||3+\sqrt[3]{4}||_{\sqrt[3]{2}}\times||3+\sqrt[3]{4}||_{1+\sqrt[3]{2}}\times||3+\sqrt[3]{4}||_{1+\sqrt[3]{4}}\times||3+\sqrt[3]{4}||_{1+2\sqrt[3]{2}-\sqrt[3]{4}}\times\\&\quad||3+\sqrt[3]{4}||_{7}\times||3+\sqrt[3]{4}||_{3+\sqrt[3]{4}}\times||3+\sqrt[3]{4}||_{1-2\sqrt[3]{2}-\sqrt[3]{4}}\times||3+\sqrt[3]{4}||_{1-2\sqrt[3]{4}}\times\cdots\\&\quad\times||3+\sqrt[3]{4}||_{\infty}\times||3+\sqrt[3]{4}||_{\infty'}\\&=1\times1\times1\times1\times1\times\frac{1}{31}\times1\times1\times\cdots\times(3+\sqrt[3]{4})\times(9+2\sqrt[3]{2}-3\sqrt[3]{4})=1\end{aligned}$$

$$\begin{aligned}&||31||_{\sqrt[3]{2}}\times||31||_{1+\sqrt[3]{2}}\times||31||_{1+\sqrt[3]{4}}\times||31||_{1+2\sqrt[3]{2}-\sqrt[3]{4}}\times\\&\quad||31||_{7}\times||31||_{3+\sqrt[3]{4}}\times||31||_{1-2\sqrt[3]{2}-\sqrt[3]{4}}\times||31||_{1-2\sqrt[3]{4}}\times\cdots\\&\quad\times||31||_{\infty}\times||31||_{\infty'}\\&=1\times1\times1\times1\times1\times\frac{1}{31}\times\frac{1}{31}\times\frac{1}{31}\times\cdots\times31\times962=1\end{aligned}$$