A Few Reflections on Noncommutative Algebraic Geometry

The Quest for the Holy Scheme

(Ninth version)

This document is a purely informal set of notes and reflections on noncommutative algebraic geometry. It is the ninth version, and contains many parts simply cut-and-pasted from the seventh and eigth versions.

Before I say anything more, there are two things I should make clearer: first, what I mean by "noncommutative algebraic geometry", and second, what these notes are supposed to be.

Noncommutative algebraic geometry is not a theory. It is an idea, and it is a quest. The quest is to make the idea into a theory; to write a book called "Elements of Noncommutative Algebraic Geometry"; to provide a noncommutative analogue of Grothendieck's work; to define a mythical object called a "skew scheme". A skew scheme is the noncommutative analogue of a scheme* (or perhaps of something more general, such as an algebraic stack). What a skew scheme is nobody knows, but it is possible to say some things about them, and that is what I intend to do: one can acquire some intuition about skew schemes and predict a few things about them, and one can also write down conditions which it is desirable that skew schemes should satisfy. This is what I intend to do.

The idea behind noncommutative geometry is the following: the second half of the twentieth century, and in particular the works of such people as Gel'fand and Grothendieck, have shown that a geometrical "space" is "essentially" the (algebra/ring/sheaf/whatever of) functions that live on that space. So the idea of noncommutative algebraic geometry is to try to build more general spaces whose algebra (or whatever) of functions would be noncommutative, and to do geometry with them. Such are skew schemes.

Grothendieck's language of schemes shows that *any* commutative ring A can be viewed as the ring of regular functions on some geometrical object, the spectrum $\operatorname{Spec} A$ of A. Such objects are affine schemes, they come with a topology called the Zariski topology, and general (commutative) schemes are obtained by gluing affine schemes together using the Zariski topology. But *affine* schemes are essentially "the same thing" as commutative rings (only morphisms go in the other direction). So it is rather natural to define an affine skew scheme to be a (noncommutative) ring: we define the category of affine skew schemes to be the opposite category to that of rings, and for any ring R we write $\operatorname{Spec} R$ and call spectrum of R the affine skew scheme that is associated to R. This step is completely trivial and we haven't done anything: the difficult part is doing geometry with these affine skew schemes and using them to define general skew schemes (perhaps not simply by gluing them: we shall see that there are reasons to think that the category of affine skew schemes is in many ways too small).

But before we say anything more, I wish to describe intuitively what the spectra of a few common rings look like, geometrically. This should serve to motivate the conditions

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* I chose the name "skew scheme" to designate the noncommutative generalization of a scheme, rather than "noncommutative scheme"; there are several reasons for that: one is that "noncommutative scheme" is too long; also, it is a bit unappropriate, since it is really the sheaves and rings that are noncommutative, not the scheme itself; finally, the word "skew" is more geometric and suggests something strange, and perhaps fuzzy.

which I will later on demand of skew schemes. It should also persuade the reader that there is something interesting to be studied, that this is not all abstract nonsense (or, worse even, just plain nonsense).

Let k be a field. Those who like to stick to concrete things will take $k = \mathbb{C}$ of course. We call $\mathbb{M}_n(k)$ the ring of $n \times n$ matrices with coefficients in k and $\mathbb{T}_n(k)$ the subring of $\mathbb{M}_n(k)$ consisting of those matrices which are upper-triangular (in the weak sense of course — all our rings have a unit). The (affine) skew schemes $\operatorname{Spec} \mathbb{M}_n(k)$ and $\operatorname{Spec} \mathbb{T}_n(k)$ will turn out to be rather interesting objects.

We start with $X = \operatorname{Spec} \mathbb{T}_n(k)$. I argue that it has n points (as we will point out later, it doesn't always make sense to talk of the points, or even of the number of points, of an affine skew scheme, but in this case it does). To see this, let $I = \operatorname{Spec} k^n$. Now I is a well-known object, as it is a commutative scheme: it consists of n (closed) points. We have two evident morphisms $\varphi \colon k^n \to \mathbb{T}_n(k)$ and $\psi \colon \mathbb{T}_n(k) \to k^n$, which give two morphisms $\varphi^a \colon X \to I$ and $\psi^a \colon I \to X$ respectively, and we have $\varphi^a \psi^a = 1_I$ (because $\psi\varphi = 1_{k^n}$). Now ψ is surjective, and by analogy with (ordinary, commutative) algebraic geometry, we are therefore led to believe that ψ^a defines the inclusion of I as a closed subscheme of X. But on the other hand, ψ writes k^n as the quotient of $\mathbb{T}_n(k)$ by a nilpotent ideal (strictly upper-triangular matrices) which tends to make us believe that X contains very little more than I. Another reason why X should have n points is that if $f \in \mathbb{T}_n(k)$ is a "regular function" on X then we can actually calculate the values of f on each of the n points: they are simply the values down the diagonal of f. And indeed these are the values taken by $\psi(f)$ on the scheme I. Values taken by f are therefore simply eigenvalues of f, which is reasonable because λ is an eigenvalue of f iff $f - \lambda$ is not invertible, a satisfactory definition for "f takes the value λ".

Now what does X have apart from just n points? It has nilpotent elements. The commutative example of $\operatorname{Spec} k[\varepsilon]$ with $\varepsilon^2 = 0$ (the so-called "dual numbers") shows that nilpotent elements add not new points but "thickness" and "texture" to existing points. So X is n points with extra texture. Now let $\mathbb{G}_n(k)$ stand for the subring of $\mathbb{T}_n(k)$ consisting of matrices which take constant values on each of the diagonals, that is, matrices of the form

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ 0 & a_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_2 \\ 0 & \cdots & 0 & a_1 \end{pmatrix}$$

The ring $\mathbb{G}_n(k)$ is commutative, and its spectrum is just one point with lots of thickness. Note also that the dimension of $\mathbb{G}_n(k)$ over k is n; now if a ring of dimension n is to have just one point, it seems reasonable that a ring of dimension n^2 (or $\frac{n(n+1)}{2}$) should have n points. The canonical morphism $\mathbb{G}_n(k) \to \mathbb{T}_n(k)$ corresponds to a morphism $\operatorname{Spec} \mathbb{T}_n(k) \to \operatorname{Spec} \mathbb{G}_n(k)$ that collapses all n points to a single one but still preserves the texture provided by the nilpotent elements.

All this is well and good, but where does the noncommutativity come into play? The point is that though the location of the points is well-defined, that of the infinitesimal part is impossible to grasp: for example in the case $n = 2$ we have only one nilpotent matrix (up to a scalar), which must correspond to an almost zero function, but where? Multiply it by the function which is one on the first point and zero on the second (i.e. the diagonal matrix with diagonal values $(1, 0)$) and you don't change it if you multiply on the left whereas you cancel it you multiply it on the right. So the infinitesimal part in question is not really near the first point nor near the second, it is somehow near both at once; the way I see it, it forms a kind of "bridge" between them. To summarize, $\operatorname{Spec} \mathbb{T}_n(k)$ consists of n closed, no nonsense, points, plus $\frac{n(n-1)}{2}$ "ethereal bridges" of "infinitesimal glue" between them.

Now we move on to $Y = \operatorname{Spec} \mathbb{M}_n(k)$. This ring is much more strongly noncommutative: the noncommutativity does not restrict its role to a little bit of fuzz around the points, but there is actually some magic going on. First of all, I argue that Y still has n points. Indeed, we can still calculate the eigenvalues of a matrix $f \in \mathbb{M}_n(k)$, and there are still n of them (with multiplicity, at least as long as k is algebraically closed). The important difference is that this time the points cannot be distinguished: it does not mean anything to ask which eigenvalue is which. This is why I say that the n points are constantly moving (this is figurative of course). Or in other words, $Y = \operatorname{Spec} \mathbb{M}_n(k)$ is a mess, but taking a snapshot of it (such as the morphism $Y \to X$ that is deduced from the evident morphism $\mathbb{T}_n(k) \to \mathbb{M}_n(k)$) always reveals n points, so we say that Y has n points. This is not completely true either because on the other hand there are an infinite number of possible "places to be" (an infinite number of matrices with one eigenvalue equal to one and the others equal to zero); but "at one given time" only n points can exist "simultaneously". Ah, how nice it would be if all this could be made rigorous.

A remarkable property of $\mathbb{M}_n(k)$ is that it is simple (it has no non trivial two-sided ideals). This means in particular that any morphism from $\mathbb{M}_n(k)$ to a ring is either zero or injective. The corresponding statement for affine skew schemes is that any morphism from an affine skew scheme to $Y = \operatorname{Spec} \mathbb{M}_n(k)$ is either the inclusion of the empty set in Y or is dominant (has an image dense in Y — whatever that means). In particular, Y can have no non trivial affine skew subscheme. I interpret this intuitively by saying that affine skew subschemes do not manage to "follow the flow" in the turmoil that is going on inside Y (perhaps this is getting a little too poetic, is it?). However, Y should clearly have skew subschemes, if only open ones. For example, if $f \in \mathbb{M}_n(k)$ is neither nilpotent nor invertible, and we define a morphism $k[t] \to \mathbb{M}_n(k)$ by $t \mapsto f$, we are defining a corresponding morphism $Y \to \operatorname{Spec} k[t] = \mathbb{A}^1_k$ where $\mathbb{A}^1_k$ is the affine line over k; the image of this morphism in $\mathbb{A}^1_k$ should be the n (at most) points which are eigenvalues of f. But then let us consider the inverse image $D(f)$ of $\mathbb{A}^1_k \setminus \{0\} = \operatorname{Spec} k[t, t^{-1}]$ in Y: that should be an open skew subscheme of Y in some way (as it is the inverse image of an open subscheme of $\mathbb{A}^1_k$), namely the "domain of definition" of f, and it should be neither empty (because f is not nilpotent) nor Y (because f is not invertible). Thus $D(f)$ should be a proper open subscheme of Y, and so it is not affine. We shall come back to this example frequently, and it has numerous consequences. One is that a skew scheme is not simply obtained by gluing affine skew schemes together, but rather *pieces* of affine skew schemes. In the commutative case, an open subscheme of an affine skew scheme is not necessarily affine, but at least open subschemes form a basis for the Zariski topology — this is no longer true in the noncommutative case. Now perhaps, one may argue, is it not true that $D(f)$ is non empty; that, however, is not a tenable position, because if $D(f)$ is empty, then $D(f-\lambda)$ must also be for every eigenvalue of f; but $D(f-\lambda)$ is the inverse image of $\mathbb{A}^1_k \setminus \{\lambda\}$, and these (indeed, any two of these) cover $\mathbb{A}^1_k$, so that their inverse images must cover Y. We shall say more about this later.

One example of an affine skew scheme is of interest to physicists, and that is the example of $\operatorname{Spec} k[q,p]$ with $pq-qp=1$ (of course, they are interested in the case $k=\mathbb{C}$). This is the *phase space* of quantum mechanics, also called the Heisenberg plane, and $k[q,p]$ is the Weyl algebra. Quantum mechanics teaches us that it is not possible to precisely define a point but only a cell of measure 1 (it is not possible to measure position and momentum accurately simultaneously, but only to within accuracies satisfying Heisenberg's uncertainty relation). Unfortunately, it is not at all clear how to formalize this — but let us at least keep in mind that there is no notion of a point, and this is in accordance with the fact that there is no morphism of skew schemes $\operatorname{Spec} k \to \operatorname{Spec} k[q,p]$. A variant of this example is the *Manin plane* defined as $\operatorname{Spec} k[u,v]$ with $uv = qvu$ where $q \in k$. The most nearly commutative case is that where $q^n = 1$ for some $n \in \mathbb{N}^*$, as it is then possible to "measure" simultaneously u^n and v (in other words, they commute) or u and v^n, and we have a morphism $\operatorname{Spec} k[u,v] \to \operatorname{Spec} k[u^n, v]$ which looks suspiciously like an n-fold covering.

A wilder example of non commutativity is that of the "free plane" $\operatorname{Spec} k\langle x,y\rangle$ where x and y satisfy *no* relation. The free plane contains the ordinary (commutative) plane $\mathbb{A}^2_k = \operatorname{Spec} k[x,y]$ as well as the Heisenberg and Manin planes as closed skew subschemes (because the corresponding rings are quotients). It is thus very large. We can also ask some rather strange questions such as, what is the intersection of the commutative and Heisenberg planes in the free plane (by intersection we really mean fiber product). In this case, the answer is, empty (essentially because one will have a hard time finding a point (x,y) in the ordinary plane that satisfies $xy-yx=1$). However, the intersection of the commutative and Manin planes is the commutative scheme $\operatorname{Spec} k[x,y]/(xy-qyx) = \operatorname{Spec} k[x,y]/((1-q)xy)$ which is a closed subscheme of $\mathbb{A}^2_k$ in the evident way, namely the union of the two coordinate axes if $q \neq 1$ and the whole plane (luckily!) if $q=1$.

Now if we recapitulate what we have seen, one thing is obvious: skew schemes have no points; or, as the motto goes, *Noncommutative geometry is pointless geometry*. In particular, skew subschemes will not be defined as topological spaces with some extra structure on them (an atlas or a structure sheaf). On the other hand, the concepts of closed and open skew subschemes seem to have a meaning. This could imply that there may be an interesting functor from the category of skew schemes to the category of locales: a locale or "pointless space" is defined by its open sets, these forming a poset with finite meets and arbitrary joins, meet being distributive over join; and a map of locales is a map in the other direction of the underlying posets that preserves the meet and join operations. Thus, the category of locales is the opposite category to the category of posets with finite meets and arbitrary joins, meet distributive over join (sometimes called "frames").

Perhaps, however, the topology of skew schemes need be more complicated even than a locale. The appropriate notion is probably that of a Grothendieck topology, or site. This is already a familiar to algebraic geometers: whereas a scheme is a topological space for the Zariski topology, the really interesting topologies (étale, fppf, fpqc and so on) are Grothendieck topologies.

In fact, the topology is essentially the only thing we need to know to build schemes: classically a scheme is defined as a locally ringed space which is locally isomorphic to the spectrum of a ring, but it may also be defined as a sheaf of sets on the category of affine schemes, which is obtained by gluing affine schemes together, for the Zariski topology. Replacing the Zariski topology by the bigger topologies (étale, fppf, fpqc), and possibly also replacing sheaf of sets by stack of groupoids (which amounts to allowing groups of automorphisms), we get diverse generalisations of schemes: algebraic spaces of Artin & Knutson, algebraic stacks of Deligne & Mumford or Artin. It is tempting to define skew schemes in a similar way, but we encounter two difficulties, which we now present.

First, in the commutative theory, the notion of *flatness* plays an essential role. Indeed, the morphisms constituting the various topologies are all flat (fpqc: flat and quasi-compact; fppf: flat and finitely presented; étale: flat, finitely presented and unramified; Zariski: flat, finitely presented, unramified and universally injective). Unfortunately, there is (as far as I know) no satisfying notion of flatness for noncommutative rings.

But there is another, more serious, problem: all the Grothendieck topologies we mentioned (and indeed, any interesting Grothendieck topology) are subcanonical (this means that representable presheaves are sheaves — or that the topology is smaller than the canonical topology, which consists of universally effective epimorphic families, and is the biggest topology for which representable presheaves are sheaves). In fact, they are obtained precisely by restricting the canonical topology to certain classes of morphisms (those which are flat and quasi-compact for the fpqc topology, flat and finitely presented for the fppf topology and so on). Now the problem is that in the case of noncommutative rings (I really mean, affine skew schemes) the canonical topology is *utterly worthless*; for example, $\mathbb{A}^1_k$ is not covered by $\mathbb{A}^1_k \setminus \{0\}$ and $\mathbb{A}^1_k \setminus \{1\}$ (because the two corresponding morphisms of rings, $k[t] \to k[t, t^{-1}]$ and $k[t] \to k[t, (t-1)^{-1}]$, give zero when pushed out along the morphism $k[t] \to \mathbb{M}_n(k)$ which sends t to a matrix f that has both zero and one as eigenvalues).

This is another hint that the category of rings is in many ways inadequate. And here is another (or rather, the same in another form): in commutative algebraic geometry, the functor Spec from the category of commutative rings to the category of schemes has a right adjoint (which also means that it is a right adjoint — remember that these are contravariant functors): if A is a commutative ring and X is a scheme then (naturally) $\operatorname{Hom}(X, \operatorname{Spec} A) = \operatorname{Hom}(A, \Gamma(X))$ where Γ is the right adjoint in question, called the global section functor. But that implies that Spec takes colimits in the category of commutative rings to limits in the category of schemes, and in particular, tensor products to fiber products. Now if we want to do the same in noncommutative geometry, then considering the (always the same) morphism $k[t] \to \mathbb{M}_n(k)$ taking t to f where f has both zero and one as eigenvalues, and its pushout along the morphism $k[t] \to k[t, t^{-1}]$, we are forced to believe that the fiber product (in this case, inverse image) $D(f)$ of $\mathbb{A}^1_k \setminus \{0\}$ and $\operatorname{Spec} \mathbb{M}_n(k)$ over $\mathbb{A}^1_k$ is empty, which is not a tenable position.

Well, so, the category of rings is a bad boy. What is wrong with it? It doesn't have satisfying pushouts. In fact, a very characteristic fact is that although these pushouts exist, they do not even have a classical name. This is also related to the problem with quotients (because quotients are particular cases of pushouts): one can quotient a noncommutative ring only by a two-sided ideal; in other words, when in a ring R one wants to make $f = 0$

(in other words, if R is a k-algebra (k not necessarily a field; for example take $k = R$) take the pushout of the morphism $k[t] \to k$ making $t = 0$ along the morphism $k[t] \to R$ sending t to f) one is forced to annihilate the entire two-sided ideal generated by f. That is too much: essentially, we want $R/(f)$ to be zero iff f is invertible (that is, on the left *and* on the right).

The way I see it, affine skew schemes are a little like G-sets: there is a kind of flow, action, turmoil going on inside them. Looking only at affine skew schemes is like looking only at G-sets: one sees only the saturated parts of existing objects, and an object like $\operatorname{Spec} \mathbb{M}_n(k)$, in analogy with a transitive G-set, has no strict parts. To make the analogy even more vivid, consider the following category: its objects are pairs (G, X) with G a group and X a G-set, and a morphism $f\colon (G, X) \to (G', X')$ is a pair $(f^\natural, f_\sharp)$ where $f^\natural\colon G' \to G$ is a morphism of groups and $f_\sharp\colon X \to X'$ is a map of sets which satisfies $f_\sharp(f^\natural(g') \cdot x) = g' \cdot f_\sharp(x)$ for any $x \in X$ and $g' \in G'$. Now if G is a group and P a sub*set* of G then we have evident objects (G, G) (where G acts on itself by left multiplication) and $(1, P)$, both with a morphism going to $(1, G)$. Now the fiber product of these objects is (G, P_G) where $P_G = G$ if $P = G$ and $P_G = \varnothing$ otherwise. We omit the easy proof. Of course, the "correct" fiber product is (G, P), and if we want to obtain this, we have to enlarge the category. This is also what we propose to do for affine skew schemes (or equivalently, rings); only I don't know how to proceed.

One more point in the same style: if R is a ring and $e \in R$ is an idempotent, we want e to "cleave" $\operatorname{Spec} R$ in two just like it does in the commutative case. But the formula $R = R/(e) \times R/(1-e)$ works only for *central* idempotents. In the geometric language described above, this is because that is exactly the condition for the two parts of $\operatorname{Spec} R$ determined by e (the locus of zeroes and the locus of ones of e) to be stable under the "internal turmoil" of $\operatorname{Spec} R$. But it should always be true that, as skew schemes, $\operatorname{Spec} R$ is the disjoint union (coproduct) of $\operatorname{Spec} R/(e)$ and $\operatorname{Spec} R/(1-e)$ where $R/(e)$ is the quotient of R by e not in the category of rings but in the generalized category that we are seeking.

Now it is time to give names to things that do not exist of yet. The generalisation of rings I propose to term "nibelungen", as they are somewhat mysterious and mythical (if not mystical). The corresponding generalisation of affine skew schemes I propose to call "naffine skew schemes" (the "n" for "nibelungen" of course — and then it will be hard to distinguish between "an affine skew scheme" and "a naffine skew scheme", thus making noncommutative algebraic geometry seminars easier to give). It is intuitively clear that the inclusion of the category of rings in the category of nibelungen should have a left adjoint (essentially because nibelungen should be constructed by taking pushouts of rings and we can always revert to the ordinary, if stupid, pushout in the category of rings). Thus we can speak of the ring of a nibelung. And the fact that this left adjoint is not very pleasant in many ways can be said as follows: the ring of the nibelungen is accursed.

Now that I have had my little laugh, here are the promised conditions that we want skew schemes and nibelungen to satisfy:

Conditions. *The categories of (noncommutative) rings, of nibelungen and of skew schemes satisfy the following conditions:*

(i) The category of rings is (up to equivalence) a full subcategory of the category of

nibelungen (and shall be identified as such).

(ii) *The inclusion of the category of rings in the category of nibelungen has a left adjoint.*

(iii) *The category of nibelungen has arbitrary (small) limits and finite colimits (the later are called "tensor products").*

(iv) *Given any reasonable definition, a "commutative nibelung" is the same thing as a commutative ring.*

(v) *If A, B, C are commutative rings then the tensor product of B and C over A is the same when A, B, C are considered as commutative rings as when they are considered as nibelungen. (In other words, the inclusion functor from the category of commutative rings to the category of nibelungen preserves finite colimits — perhaps because it has a right adjoint.)*

(vi) *For any ring R and any element f of R, the nibelung $R/(f)$ (defined as the tensor product over $R[t]$ of R with itself, where the two morphisms $R[t] \to R$ take t to 0 and f respectively) is zero iff f is (left and right) invertible in R.*

(vii) *The category of schemes is (up to equivalence) a full subcategory of the category of skew schemes (and shall be identified as such).*

(viii) *The opposite category of the category of rings, namely the category of affine skew schemes, is (up to equivalence) a full subcategory of the category of skew schemes (and we shall write* Spec *for the contravariant inclusion functor of the category of rings in the category of skew schemes).*

(ix) *The opposite category of the category of nibelungen, namely the category of naffine skew schemes, is (up to equivalence) a full subcategory of the category of skew schemes (and we shall still write* Spec *for the contravariant inclusion functor of the category of nibelungen in the category of skew schemes).*

(x) *The functor* Spec *from the category of nibelungen to the category of skew schemes has a right adjoint (which we call the global section functor and which we write Γ).*

(xi) *The category of skew schemes has finite limits and small coproducts (perhaps even small colimits).*

(xii) *The category of skew schemes comes with a topology, which induces on the category of schemes one of the classical topologies, viz. Zariski, étale, fppf or fpqc.*

(xiii) *The topology mentioned under item (xii) is subcanonical; more than that, it is obtained by restricting the canonical topology of the category of skew schemes to a certain class of morphisms.*

(xiv) *For the topology mentioned under item (xii), any skew scheme is covered by naffine skew schemes.*