A Few Reflections on Noncommutative Algebraic Geometry
The Quest for the Holy Scheme

I here wish to present a few reflections on noncommutative algebraic geometry, which I may also call the "Quest for skew schemes". The ultimate goal is to answer the following question

Quest. *What is noncommutative algebraic geometry? What is a skew scheme*[*]*? In other words, how can noncommutative rings be glued together? Or yet again: how can we give a noncommutative analog of Grothendieck's work?*

I am certainly in no position to give a complete, or even partial answer to this Quest; as a matter of fact, I am not even capable of giving a satisfactory statement of what the Quest actually means. The Quest may take generations to complete; it may turn out that the "Elements of Noncommutative Algebraic Geometry" are hundreds of thousands of pages in size; it may even turn out that there is no such thing as noncommutative algebraic geomtry. But then, it may also turn out that only very little is needed to fulfill the Quest and provide us with loads of wonderful new results.

I must warn that I am no expert on the subject; I am, in fact, no expert on any subject. I barely know what a Grothendieck topology is, for example. And I know absolutely nothing of the works that others may have done toward the fulfilling of the Quest; I know next to nothing on quantum and Hopf algebras or noncommutative differential geometry (an altogether better known domain of mathematics than noncommutative algebraic geometry, see Connes (1990)). Also, there are probably many mistakes in what follows, even in those bits that do make sense; I still hope that I haven't entirely overlooked an important phenomenon. Still, I have a few ideas that I would like to set down, and also a little bit of intuition on what $\operatorname{Spec} \mathbb{M}_n(k)$ ought look like.

One thing I must specify however: the Quest is to give a very general and very broad setting for noncommutative algebraic geometry, a whole language on how to glue noncommutative rings together. Just understanding a few examples of noncommutative rings in the geometric perspective is not sufficient, illuminating though these examples may be (in particular, $\mathbb{M}_n(k)$, $k[q,p]$ with $qp - pq = 1$, or $k[u,v]$ with $vu = quv$). Some may eschew such generality, but I strongly believe that it is necessary if we are to really understand something of what is going on.

(...and Grothendieck is His Prophet.)

Before we go on, let us state certain properties that we would want noncommutative algebraic geometry to satisfy:

Properties (first part). *The category of skew schemes satisfies the following properties:*

(i) It admits arbitrary (small) inductive limits and finite projective limits.

(ii) It has the category of schemes ("commutative schemes") as a full subcategory.

(iii) It has the opposite category of (noncommutative) rings as a full subcategory, namely "affine skew schemes". (We will write Spec *for the contravariant inclusion functor from the category of noncommutative rings to the category of skew schemes.)*

Condition (i) is reasonable because it is satisfied by the categories of schemes and the opposite category of noncommutative rings; also, because the coproduct should cor-

[*] I chose the name "skew scheme" to designate the noncommutative generalization of a scheme, rather than "noncommutative scheme"; there are several reasons for that: one is that "noncommutative scheme" is too long; also, it is a bit unappropriate, since it is really the sheaves and rings that are noncommutative, not the scheme itself; finally, the word "skew" is more geometric and suggests something strange, and perhaps fuzzy.

respond to a disjoint union, which we certainly want to exist, and because the fibered product (pullback) is undoubtedly an important construction. Condition (ii) is reasonable because we want noncommutative algebraic geometry to specialize to ordinary algebraic geometry. Condition (iii) is reasonable because it mimics the corresponding statement for commutative schemes and commutative rings; and we want skew schemes to be to noncommutative rings what schemes are to commutative rings.

However, we do *not* demand, for example, the existence of some kind of (forgetful) functor to the category of topological spaces. That would be a most bold requirement because, if quantum mechanics* has taught us one thing about noncommutative geometry, it is that it has no points. As the well-known motto says, "Noncommutative geometry is pointless geometry". In fact, that is quite in agreement with the fact that noncommutative rings have too few prime ideals, and in fact too few (two-sided) ideals altogether (for example, the ring $\mathbb{M}_n(k)$ of $n \times n$ matrices with coefficients in a field k is simple, i.e. has no non trivial two-sided ideal). It may well be that there exists an interesting functor from the category of skew schemes to the category of locales (the category of locales is the opposite category of frames, and a frame is a poset with finite meets and arbitrary joins satisfying the distributivity of the meet over the join, with a morphism of frames being one which preserves both the meet and join operations; a locale ought to be thought of as some kind of "pointless space" and the corresponding frame is the set of its open sets), but locales do not always have enough points (a locale is said to have enough points iff its corresponding frame is the frame of open sets on some (sober) topological space; we refer to MacLane (1991) for a discussion on locales and localic topoi).

Note that though skew schemes do not have points in general, there are still circumstances in which it is possible to count the number of points. For example, we will give some evidence showing that $\operatorname{Spec}\mathbb{M}_n(k)$ has (presumably) n points.

In fact, we proceed immediately to study what we can say about the spectra of k^n, $\mathbb{T}_n(k)$ and $\mathbb{M}_n(k)$. Here, k is a field, and $\mathbb{T}_n(k)$ and $\mathbb{M}_n(k)$ are the rings of $n \times n$ upper-triangular and $n \times n$ matrices respectively with coefficients in k. We have canonical inclusions

$$k^n \hookrightarrow \mathbb{T}_n(k) \hookrightarrow \mathbb{M}_n(k)$$

which must therefore correspond to maps of skew schemes

$$\operatorname{Spec}\mathbb{M}_n(k) \to \operatorname{Spec}\mathbb{T}_n(k) \to \operatorname{Spec} k^n$$

Now for one thing we know $\operatorname{Spec} k^n$ very well: it is just the disjoint union of n (closed) points. We also know that in the commutative theory an injective morphism of rings corresponds to a dominant morphism of schemes (i.e. one whose image is dense). As for $\mathbb{T}_n(k)$ it corresponds to k^n plus a little bit of "extra fuzz", nilpotent elements. Looking at the example of $k[\varepsilon]$ with $\varepsilon^2 = 0$ ("dual numbers"), we see that nilpotent elements do not add extra points, but only add some kind of "thickness" or "texture" to the existing points. So $\operatorname{Spec}\mathbb{T}_n(k)$ still has n points. In fact, those points can still be identified: the inclusion of the i-th point in $\operatorname{Spec}\mathbb{T}_n(k)$ corresponds to the epimorphism sending an $n \times n$ upper triangular matrix to its i-th diagonal coefficient, which is also an eigenvalue. And indeed, all these morphisms fit together into one epimorphism $\mathbb{T}_n(k) \to k^n$ which corresponds to a $\operatorname{Spec} k^n \to \operatorname{Spec}\mathbb{T}_n(k)$ that should be a closed embedding in some way. Thus, $\mathbb{T}_n(k)$

* Some may wonder what QM has to do with the whole thing anyway. The point is that the phase space of QM is, or at any rate has something to do with, the spectrum of the ring $\mathbb{C}[q, p]$ with $qp - pq = 1$.

definitely has n points; the way I see it, any pair of points is bound together with some kind of "infinitesimal glue" or "bridge".

As another interesting morphism, we can look at the ring $\mathbb{G}_n(k)$ of matrices of $\mathbb{T}_n(k)$ which take a constant value on each diagonal, that is

$$\begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ 0 & a_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_1 \\ 0 & \cdots & 0 & a_0 \end{pmatrix}$$

The ring $\mathbb{G}_n(k)$ is a n-dimensional subring of $\mathbb{T}_n(k)$, and its spectrum has one point (with an infinitesimal neighborhood of order n around it); now if dimension n gives 1 point, it is reasonable that dimension n^2 (or $\frac{n(n+1)}{2}$) should give n points. Intuitively, the map $\operatorname{Spec}\mathbb{T}_n(k) \to \operatorname{Spec}\mathbb{G}_n(k)$ corresponding to the canonical map $\mathbb{G}_n(k) \to \mathbb{T}_n(k)$ is a map that collapses all n points to one, but somewhat preserves the infinitesimal fuzz around them.

The case of $\mathbb{M}_n(k)$ is definitely more complex than that of $\mathbb{T}_n(k)$. This time, the noncommutativity is not just an ethereal fuzz around the n points, but there is actually some magic going on here. First of all, I argue that $\operatorname{Spec}\mathbb{M}_n(k)$ still has no more than n points. In fact, if $M \in \mathbb{M}_n(k)$, so that we can view M as a function (morphism) from $\operatorname{Spec}\mathbb{M}_n(k)$ to $\operatorname{Spec} k[X]$ (corresponding to the morphism $k[X] \to \mathbb{M}_n(k)$ given by $X \mapsto M$), and if moreover k is algebraically closed (so we can see $\operatorname{Spec} k[X]$ as essentially k) then we can calculate the values taken by the function M. Namely, these are the eigenvalues of M. Indeed, to say that λ is an eigenvalue means that $M - \lambda$ is not invertible, and that ought certainly be the same as saying that M takes the value λ. Now an $n \times n$ matrix has at most n distinct eigenvalues, and it is therefore reasonable to think that $\mathbb{M}_n(k)$ has exactly n points. However, as opposed to the previous case, it is not possible to identify these n points: there exists no non trivial morphism $\mathbb{M}_n(k) \to k$. In fact, there exists no non trivial surjective morphism from $\mathbb{M}_n(k)$ to anything; that would seem to say that it is not possible to find any closed subscheme of $\operatorname{Spec}\mathbb{M}_n(k)$. I rather like to think that it is not possible to "fix" ("isolate", "locate", "identify", etc) a closed subscheme of $\operatorname{Spec}\mathbb{M}_n(k)$ (other than the empty and full schemes of course). Intuitively, that is because, although the n points exist, they are constantly moving in some magical way, and it is not possible to pick one out. How we are to overcome this difficulty I do not know. But I still feel that somehow $\operatorname{Spec}\mathbb{M}_{n-1}(k)$ should be somehow a subscheme of $\operatorname{Spec}\mathbb{M}_n(k)$, even though there are no interesting morphisms in either direction; perhaps that implies abandoning property (iii) above (at least the fullness assertion), but more probably there will be another (necessarily non affine) subscheme of $\operatorname{Spec}\mathbb{M}_n(k)$, that looks a lot like $\operatorname{Spec}\mathbb{M}_{n-1}(k)$ but still differs in subtle ways. To summarize the problem: the inner automorphisms of $\mathbb{M}_n(k)$ give rise to automorphisms of $\operatorname{Spec}\mathbb{M}_n(k)$ which permute the n points transitively. In general, we can predict that the existence of inner automorphisms will lead to a lot of trouble (ah, but we're asking for trouble, aren't we?).

Now we move on to the next major point, which is the covering question and the problem with pushouts.

Suppose A is a commutative ring and $a_1, \ldots, a_n$ are elements of A which generate the unit ideal. We denote by A_{a_i} the localization of A at a_i (i.e. the ring obtained by making a_i invertible in A). We have canonical morphisms $A \to A_{a_i}$. We can also make a_j invertible in A_{a_i}: that is the same as making a_i invertible in A_{a_j}, or making $a_i a_j$ invertible in A ("the same" in the sense that it leads to the same ring) and we get $A_{a_i a_j}$ this way.

We have $A_{a_i a_j} = A_{a_i} \otimes_A A_{a_j}$ (note in particular that $A_{a_i} \otimes_A A_{a_i} = A_{a_i}$: this may turn out to be important, who knows). We can consider the diagram

$$A \to \prod_i A_{a_i} \rightrightarrows \prod_{i,j} A_{a_i a_j}$$

where one of the two arrows on the right sends $(s_i)_i$ to $(s_i/1)_{i,j}$ and the other sends $(s_j)_j$ to $(s_j/1)_{i,j}$. I claim that this diagram is an equalizer, in other words that the arrow on the left is injective, and that its image is exactly the set of points where the two arrows on the right coincide. That is just part of the statement that the structure sheaf on $\operatorname{Spec} A$ is indeed a sheaf, and it corresponds to the fact that the images of the morphisms $\operatorname{Spec} A_{a_i} \to \operatorname{Spec} A$ cover $\operatorname{Spec} A$.

More generally, suppose that we have homomorphisms $A \to B_i$ of commutative rings. We would be tempted to define "the B_i cover A" to mean "the diagram

$$A \to \prod_i B_i \rightrightarrows \prod_{i,j} B_i \otimes_A B_j \tag{$*$}$$

is an equalizer". That does not work so well, however. For example, take $A = k[x, y]$, $B_1 = k[x, y]_x$, $B_2 = k[x, y]_y$: one verifies readily that the above diagram is then an equalizer, but $\operatorname{Spec} B_1$ and $\operatorname{Spec} B_2$ do not cover $\operatorname{Spec} A$ as they leave out the origin. Rather, let us say that the B_i cover A iff for every homomorphism $A \to C$ of commutative rings the diagram

$$C \to \prod_i B_i \otimes_A C \rightrightarrows \prod_{i,j} (B_i \otimes_A C) \otimes_C (B_j \otimes_A C)$$

is an equalizer. In this case, the images of the $\operatorname{Spec} B_i$ cover $\operatorname{Spec} A$; indeed, if $\mathfrak{p}$ is a prime ideal of A, then by putting $C = \varkappa_{\mathfrak{p}} = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ (the residue field), we see that $\varkappa_{\mathfrak{p}} \to \prod_i B_i \otimes_A \varkappa_{\mathfrak{p}}$ is injective, and in particular for some i we have $B_i \otimes_A \varkappa_{\mathfrak{p}}$ non zero and therefore it has a maximal ideal, so $\mathfrak{p}$ is in the image of $\operatorname{Spec} B_i$. The condition that the images of $\operatorname{Spec} B_i$ cover $\operatorname{Spec} A$ (as topological spaces) is not sufficient for the B_i to cover A in the sense defined above, since they may be omitting infinitesimal neigborhoods (for example, k alone does not cover $k[\varepsilon]$ with $\varepsilon^2 = 0$, by the map sending ε to 0, even though the two rings have the same spectrum).

Now, the important thing is that the definition of covering that we have just given puts a Grothendieck topology on the category of affine (commutative) schemes; and this is in fact straightforward to verify. Moreover, the condition that the diagram $(*)$ be an equalizer is exactly the condition stating that the functor $\operatorname{Spec} A \mapsto A$ is a sheaf. This topology is finer than the Zariski topology (which is a Grothendieck topology on the category, say, of affine commutative schemes but with open embeddings as arrows, i.e. open sets) or even the étale topology (where this time open sets are étale morphisms), in the sense that it has more open sets. However, when restricted to the category of affine schemes with open embeddings as arrows, the big topology considered above gives back the Zariski topology.

(If I may add a question at that point, what happens if we only impose the injectivity of $C \to \prod_i B_i \otimes_A C$ and we completely forget about the other half of the diagram: do we actually lose something? To make this a self-standing question, suppose $(B_i)_{i \in I}$ is an arbitrary family of A-algebras such that $C \to \prod_i B_i \otimes_A C$ is injective for every A-algebra C; does it follow that for every A-algebra C the diagram

$$C \to \prod_i B_i \otimes_A C \rightrightarrows \prod_{i,j} (B_i \otimes_A C) \otimes_C (B_j \otimes_A C)$$

is an equalizer? This is certainly the case if, say, I is finite, and for each i, $B_i = A_{a_i}$ for some $a_i \in A$.)

We can also generalize this condition to commutative schemes: if $Y_i \to X$ are morphisms of schemes, then we say that the Y_i cover X iff for every scheme Z over X the diagram

$$\coprod_{i,j}(Y_i \times_X Z) \times_Z (Y_j \times_X Z) \rightrightarrows \coprod_i Y_i \times_X Z \to Z$$

is a coequalizer (a diagram $U \rightrightarrows V \to W$ of schemes is called a coequalizer iff every morphism $V \to W'$ of schemes which equalizes the two arrows on the left factors uniquely through the arrow on the right). This way we get a Grothendieck topology on the category of schemes, and the global section functor is a sheaf on this site. (Of course, there are many verifications to be done, and I actually didn't do any of them; I just hope very much that I'm right; also, I didn't check that this condition actually generalizes the one given above — this is not obvious, since the spectrum of an infinite product is not the disjoint union of the spectra, but comes with a lot of unwanted points to make the whole thing quasi-compact.)

What have we gained by doing this? Several things. First of all, suppose we didn't know in the first place what a scheme is, but only a commutative ring. Then if we just knew what it means for $A \to B$ to correspond to an open embedding $\operatorname{Spec} B \to \operatorname{Spec} A$, we would know when affine open sets cover $\operatorname{Spec} A$, and so in fact we would know how to take the union of affine open sets, therefore we would know what an arbitrary open set is, and finally what a scheme is (first as a locale, and then as a topological space). Second, suppose X is a scheme over $\operatorname{Spec} R$ for some commutative ring R; then we may look at X as a functor from the category of R-algebras to the category of sets, which takes an R-algebra A to the set $X(A)$ of points of $X \times_{\operatorname{Spec} R} \operatorname{Spec} A$. This functor is continuous for the Grothendieck topology just defined (I hope I'm not talking too much nonsense — I hardly have an idea of what all this means). I believe this leads to something called "stacks", and this may suggest interesting possibilities for noncommutative algebraic geometry.

All this suggests that we add the following property to our category of skew schemes:

Properties (second part). *The category of skew schemes also satisfies the following properties:*

(iv) *It comes with a Grothendieck topology, which restricts to the one defined above in the case of commutative schemes, and*

(v) *with a sheaf on this site, the "global section functor", which on the full subcategory of affine skew schemes is just the functor* $\operatorname{Spec} A \mapsto A$.

(vi) *Any skew scheme can be covered by a family of affine skew schemes.*

Condition (iv) is where Geometry really comes in rather than plain algebra. Condition (v) seems almost indispensible to ensure that our objects retain some kind of algebraic reality. As for condition (vi), it is *faute de mieux* a way of saying that skew schemes should be made by patching affine skew schemes together.

This leads us to the problem of the pushouts. Consider $k[x]$ (where k is a field, say); it is covered by $k[x]_x$ and $k[x]_{x-1}$ (geometrically, this says that $\mathbb{A}^1(k) = \operatorname{Spec} k[x]$ is covered by the complement of 0 and the complement of 1). Now by the properties above and the definition of a Grothendieck topology (on a category with pullbacks) this implies that for any skew scheme X over $\operatorname{Spec} k[x]$, the pullbacks X_0 and X_1 of $\operatorname{Spec} k[x]_x$ and $\operatorname{Spec} k[x]_{x-1}$ to X over $\operatorname{Spec} k[x]$ should cover X. Geometrically, these correspond to taking the inverse images of the complements of 0 and 1 respectively by the given map $X \to \operatorname{Spec} k[x]$. That is quite reasonable. Now take $X = \operatorname{Spec} \mathbb{M}_n(k)$, and map $k[x]$ to $\mathbb{M}_n(k)$ by taking x to the matrix

$$M = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

The corresponding map $X = \operatorname{Spec} \mathbb{M}_n(k) \to \operatorname{Spec} k[x] = \mathbb{A}^1(k)$ takes one of the (indistinguishable) n points of X to 1 and the other ones to 0. Now the pushout of $k[x]_x$ (or $k[x]_{x-1}$) to $\mathbb{M}_n(k)$ over $k[x]$ is zero, which is rather unpleasant, because zero is going to have a hard time covering anything. In other words, even though the category of noncommutative rings has pushouts, these pushouts are not very interesting, and in particular are not the pullbacks in the category of skew schemes that we are looking for. Or to say things otherwise, the pullback of $\operatorname{Spec} k[x]_x$ to $\operatorname{Spec} \mathbb{M}_n(k)$ over $\operatorname{Spec} k[x]$ is not affine, contrary to what happens in the commutative case. This is actually quite reasonable, because in taking the inverse image of the complement of 0, we are trying to distinguish one of the n indistinguishable points of X. The unfortunate consequence, however, is that the global section functor is not going to have all the nice properties that one would expect of it...

Let us conclude with a little daydreaming about open sets and localizations. In the commutative case the principal open sets $D(f)$ corresponding to localization maps $A \to A_f$ form a basis of $\operatorname{Spec} A$, and the $D(f_i)$ cover $\operatorname{Spec} A$ just in case the f_i generate the unit ideal in A. Localization also exists in the noncommutative case: if A is any ring and $f \in A$, then there exists a ring A_f and a ring homomorphism $A \to A_f$ universal among those ring homomorphisms which make f invertible. It does not seem that these noncommutative localizations have nice properties; in particular, I do not know of any nice description of the kernel of $A \to A_f$, or any nice condition for $A \to A_f$ to be injective. Also, as opposed to the commutative case, multiple localizations do not necessarily reduce to one. Anyhow, as we have already pointed out, the principal open set $D(f)$, defined as the universal *skew scheme* making f invertible (whatever that means), need not be affine. Just what it means for the $D(f_i)$ to cover $\operatorname{Spec} A$ is probably a very difficult question, because it is in many ways the quintessence of noncommutative algebraic geometry. One thing we can say for certain is that generating the unit ideal as a two-sided ideal is too weak a condition (if $M \in \mathbb{M}_n(k)$ is the matrix defined above, then we don't want $D(M)$ to be all of $\operatorname{Spec} \mathbb{M}_n(k)$ — why, that matrix is zero in $n-1$ points !). Perhaps generating a one-sided ideal is better, but then one would have to distinguish between left and right coverings, which is assuredly unpleasant*.

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* Who said it had to be pleasant?