Elements of noncommutative algebraic geometry

The language of schemes

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Rings, localizations and multicalizations.

By a "ring" we mean a generally non commutative ring with a unit element, and ring homomorphisms are assumed to take the unit element to the unit element. By "invertible" we mean "left and right invertible".

If A is a ring and f an element of A, we will write A_f for the universal ring above A making f invertible; in other words, there is a canonical homomorphism $\theta: A \to A_f$ such that $\theta(f)$ is invertible, and, for any homomorphism $h: A \to B$ with $h(f)$ invertible there exists a unique morphism $\hat{h}: A_f \to B$ verifying $h = \hat{h}\theta$. The existence of A_f is easy to check; in fact, it can be constructed as follows: let A' be the ring of formal sums $a_0 x a_1 \cdots a_{n-1} x a_n$ (with the obvious multiplication), and let I be the two-sided ideal of A' generated by $1 - fx$ and $1 - xf$; then $A_f \simeq A'/I$. Uniqueness of A_f up to isomorphism follows from general nonsense. Although the canonical map $A \to A_f$ is not injective in general [and I would be happy to learn of some conditions on A and f for it to be], we will frequently identify an element $g \in A$ with its image in A_f; in particular, we do that for f, and we write f^{-1} for the inverse of f in A_f. Note that if f and g commute in A then f^{-1} and g commute in A_f — hence, if A is commutative, then so is A_f, and it is therefore the ordinary commutative localization of A at f. We shall therefore say, even when A is not commutative, that A_f is the localization of A at f.

If f and g are two elements of A, then making fg invertible is the same thing as making both f and g invertible; in other words, A_{fg} and $(A_f)_g$ (this really means $(A_f)_{\theta(g)}$ where $\theta: A \to A_f$ is the canonical map) are canonically isomorphic, and, moreover, the canonical map $A \to A_{fg}$ corresponds under this isomorphism to the composite of the canonical maps $A \to A_f \to (A_f)_g$. Therefore, all the objects A_{fg}, A_{gf}, $(A_f)_g$ and $(A_g)_f$ can be identified — and we shall do so with no further comment.

However, and this is where the analogy with the commutative case breaks down, multiple localizations do not reduce to one: in other words, if f is an element of A and g an element of A_f, it is not always true that there exists h in A such that $(A_f)_g = A_h$. We therefore define a <u>multicalization</u> of a ring A as a ring of the form $(\cdots(A_{f_1})_{f_2}\cdots)_{f_n}$, where each f_i belongs to $(\cdots(A_{f_1})_{f_2}\cdots)_{f_{i-1}}$. To simplify notation, we write this as $A_{f_1;\ldots;f_n}$, and we say that $f_1, \ldots, f_n$ forms a "multicalizing family". We have a canonical homomorphism $A \to A_{f_1;\ldots;f_n}$ obtained by composing the various canonical homomorphisms.

If $\varphi: A \to B$ is a ring homomorphism, and $f \in A$, then we have a localization of φ at f, defined in the following way: the image of f under the composite map $A \to B \to B_{\varphi(f)}$ (of φ and the canonical map) is invertible, so by the universal property of A_f, it lifts to a map $A_f \to B_{\varphi(f)}$ which we note φ_f (or simply φ) when there is no ambiguity. We have the functoriality conditions $(\mathrm{id}_A)_f = \mathrm{id}_{A_f}$ and $(\psi\varphi)_f = \psi_{\varphi(f)}\varphi_f$; moreover, the canonical map $A \to A_g$ localizes to the canonical map $A_f \to A_{fg}$ (defined above).

The same applies to multicalizations: if $f_1, \ldots, f_n$ is a multicalizing family for a ring A and $\varphi: A \to B$ is a homomorphism, then we have a homomorphism $A_{f_1;\ldots;f_n} \to B_{\varphi(f_1);\ldots;\varphi(f_n)}$ obtained by successively localizing φ at the various f_i.

Principal and semiprincipal open sets.

We formally write $D(f) \subseteq D(g)$ (or $D(g) \supseteq D(f)$) iff g is invertible in A_f (meaning, of course, that the image of g under the canonical map $A \to A_f$ is invertible in A_f); in other words, iff there is a "canonical" map $A_g \to A_f$. When this holds, we say that $D(f)$ is contained in $D(g)$. We write $D(f) = D(g)$ iff $D(f) \subseteq D(g)$ and $D(g) \subseteq D(f)$, in which case we have a canonical isomorphism $A_g \simeq A_f$. Such a $D(f)$ (that is to say, the class of f for the equivalence relation just defined) is called a principal open set of $\operatorname{Spec} A$ [note that we are not assigning any meaning to $\operatorname{Spec} A$ yet]; when there is any ambiguity on the ring in question, we will write $D_A(f)$. We further let $\varnothing = D(0)$ (the empty set), and $\operatorname{Spec} A = D(1)$ (the full set). For every $f \in A$ we have $\varnothing \subseteq D(f) \subseteq \operatorname{Spec} A$. Thus, the principal open sets, partially ordered by $\subseteq$, have a greatest and a least element, namely $\varnothing$ and $\operatorname{Spec}(A)$ (moreover, as soon as A is not the zero ring, these are distinct).

If $f, g \in A$, then it follows from the general remarks we made on localizations that $D(fg) = D(gf)$. Moreover, $D(fg)$ is the largest principal open set contained in both $D(f)$ and $D(g)$ (because making fg invertible is the same as making both f and g invertible). We therefore define $D(f) \cap D(g) = D(fg)$, and we say that $D(fg)$ is the intersection of $D(f)$ and $D(g)$. Note that $\cap$ is commutative and associative, and that $\varnothing \cap D(f) = \varnothing$ and $D(f) \cap D(f) = D(f)$ for every $f \in A$ (the latter because $D(f^k) = D(f)$ for every k).

We note for future reference that if $\varphi\colon A \to B$ is a ring homomorphism, then the map $\varphi^{*\natural}$ (this notation will be explained later) taking $D(f)$ to $D(\varphi(f))$ is well defined and preserves intersection.

For $f \in A$, we now define $\Gamma(D(f), \mathcal{O}_A) = A_f$; when the context is clear, we will sometimes write this $\mathcal{O}_A(D(f))$ or $\Gamma(D(f))$. Note that this definition makes sense, because, as we pointed out, if $D(f) = D(g)$ then $A_f \simeq A_g$. If $D(f) \subseteq D(g)$ then we have a canonical map $\rho^{D(g)}_{D(f)}\colon \Gamma(D(g)) \to \Gamma(D(f))$ which we call a restriction map (this has already been mentioned; the map in question is obtained by lifting the canonical map $A \to A_f$ to a map $A_g \to A_f$ by using the universal property of A_g). Restriction maps satisfy the pleasant properties that $\rho^{D(f)}_{D(f)} = \mathrm{id}_{\Gamma(D(f))}$ and, if $D(f) \subseteq D(g) \subseteq D(h)$ then $\rho^{D(h)}_{D(f)} = \rho^{D(g)}_{D(f)} \rho^{D(h)}_{D(g)}$. In Grothendieckian terms, we have a presheaf on the category of principal open sets of $\operatorname{Spec} A$ with value in the category of (non commutative) rings.

Now, we certainly want this presheaf (which eventually will turn out to be the structure sheaf on $\operatorname{Spec} A$) to be a sheaf. In order to do that, we need a Grothendieck topology; that is, we need to be able to say when principal open sets $D(f_i)$ cover a principal open set $D(f)$ that contains them. The first reflex is to take the minimal requirement for $\mathcal{O}_A$ to be a sheaf, that is, postulate that the $D(f_i)$ cover $D(f)$ iff the canonical map $A_f \to \varprojlim A_{f_i}$ (to be detailed in a moment) is an isomorphism. That is too weak a condition, however: it fails to satisfy a lot of simple nice properties that we would want coverings to satisfy, and in particular fails to define a Grothendieck topology [well, I have not found any counterexample, but I'm pretty sure it must fail]. We shall say that the $D(f_i)$ quasi-cover $D(f)$ when the previous condition is satisfied; and we say that they cover $D(f)$ iff for every ring homomorphism $\varphi\colon A \to B$ the $D(\varphi(f_i))$ quasi-cover $D(\varphi(f))$.

But before we proceed further in that direction, we will introduce semiprincipal open sets. The necessity for these arises from a different concern. Namely, when we define skew schemes, we will want an open set of a skew scheme to be a skew scheme, and for that we will need a principal open set of a principal open set to be a principal open set; that will not be true unfortunately, because as we already pointed out the localization of a localization is not always a localization. However, things will work nicely if instead of using principal open sets, based on localizations, we use semiprincipal open sets, based on multicalizations.

Therefore, we now consider objects $D(f_1; \ldots; f_n)$ where $f_1, \ldots, f_n$ is a multicalizing family. We shall say $D(f_1; \ldots; f_n) \subseteq D(g_1)$ iff g_1 is invertible in $A_{f_1;\ldots;f_n}$, in which case we have a canonical map $A_{g_1} \to A_{f_1;\ldots;f_n}$. Then it makes sense to ask whether a $g_2 \in A_{g_1}$ is invertible in $A_{f_1;\ldots;f_n}$. When that is so, we shall say $D(f_1;\ldots;f_n) \subseteq D(g_1;g_2)$. And in general, we shall say that $D(f_1;\ldots;f_n) \subseteq D(g_1;\ldots;g_m)$ iff every g_j is invertible in $A_{f_1;\ldots;f_n}$, when considered as an element of that ring using the canonical map $A_{g_1;\ldots;g_{j-1}} \to A_{f_1;\ldots;f_n}$ that exists because of the previous relation. To make a long story short, we say that $D(f_1;\ldots;f_n) \subseteq D(g_1;\ldots;g_m)$ iff there exists a canonical map $A_{g_1;\ldots;g_m} \to A_{f_1;\ldots;f_n}$. Equality between these <u>semiprincipal open sets</u> is then defined just as we did for principal open sets. And it is clear that principal open sets can be identified with certain semiprincipal open sets (pedantically: principal open sets form a full subcategory of the category of semiprincipal open sets). The intersection can also be defined with no problem: the intersection of $D(f_1;\ldots;f_n)$ and $D(g_1;\ldots;g_m)$ is $D(f_1;\ldots;f_n;g_1;\ldots;g_m)$. We define $\Gamma(D(f_1;\ldots;f_n),\mathcal{O}_A)$ (with the same abuses of notations as were mentioned previously; this shall not be said again) to be $A_{f_1;\ldots;f_n}$. Restriction maps are defined in the obvious way and satisfy the same properties as previously.

In what follows, we shall frequently introduce new notions for/using principal open sets before doing so for/using semiprincipal open sets. The reason for doing so is not so much that the semiprincipal case is more complicated as that its notations are much more unwieldly.

General open sets.

The open sets we have defined up to now (principal and semiprincipal) allow us to take intersections but not unions. We now proceed to define general open sets (a category having arbitrary colimits and finite limits rather than just finite limits as the category of (semi)principal open sets).

We first note that if $(D(f_i))_{i\in I}$ is a family of principal open sets (that is nothing more than saying that $(f_i)_{i\in I}$ is a family of elements of A), then we have a ring $\varprojlim A_{f_i}$. Its elements are matching families of elements of the A_{f_i}, that is, families (x_i) with $x_i \in A_{f_i}$ for every i, satisfying $\rho^{D(f_i)}_{D(f_i)\cap D(f_j)}(x_i) = \rho^{D(f_j)}_{D(f_i)\cap D(f_j)}(x_j)$ for every i,j, addition and multiplication being defined term by term. We formally define the symbol $\bigcup D(f_i)$ (equality between these symbols will be defined below) and let $\Gamma(\bigcup D(f_i),\mathcal{O}_A) = \varprojlim A_{f_i}$.

If $(D(f_i))_{i\in I}$ and $(D(f_j))_{j\in J}$ are two families, the latter contained in the former, that is, $I \subseteq J$, then we have a canonical map $\varprojlim A_{f_j} \to \varprojlim A_{f_i}$. If that canonical map is an isomorphism, then we shall say that $\bigcup D(f_i)$ and $\bigcup D(f_j)$ are <u>quasi-equal</u>. We say that $\bigcup D(f_i)$ and $\bigcup D(f_j)$ are <u>equal</u> iff $\bigcup D(\varphi(f_i))$ and $\bigcup D(\varphi(f_j))$ are quasi-equal for every ring homomorphism $\varphi\colon A \to B$. We now remove the hypothesis that $I \subseteq J$: we shall say that $\bigcup_{i\in I} D(f_i)$ <u>contains</u> $\bigcup_{j\in J} D(f_j)$ iff it is equal to $\bigcup_{j\in I\cup J} D(f_i)$, in the sense just described. We write this $\bigcup_{i\in I} D(f_i) \supseteq \bigcup_{j\in J} D(f_j)$. Finally, when $\bigcup D(f_i)$ is both contained in and contains $\bigcup D(f_j)$, then we say that they are equal.

All these definitions carry over *mutatis mutandis* to the case of semiprincipal open sets: if $(U_i)_{i\in I}$ is a family of semiprincipal open sets, then we define $\varprojlim \Gamma(U_i)$ as the set of matching families of elements of the $\Gamma(U_i)$, and we call this $\Gamma(\bigcup U_i,\mathcal{O}_A)$; proceed so on replacing $D(f_i)$ by U_i and A_{f_i} by $\Gamma(U_i)$.

The objects $\bigcup_{i\in I} U_i$ with U_i semiprincipal open sets, that we have just defined, are called <u>general open sets</u>. We identify a semiprincipal open set U with the general open set $\bigcup_{i\in\{\bullet\}} U_i$ with $U_\bullet = U$. Using the comment which we have already made, namely that if $D(f_1;\ldots;f_n)$ is equal to (resp. contained in) $D(g_1;\ldots;g_m)$ then so is $D(\varphi(f_1);\ldots;\varphi(f_n))$ to/in $D(\varphi(g_1);\ldots;\varphi(g_m))$ for every homomorphism $\varphi\colon A \to B$, it requires no more than a second's thought to see that two semiprincipal sets are equal (resp. contained one in another) iff they are so as general open sets. (Pendantically, the semiprincipal open sets form a full subcategory of the category of general open sets.)

If $U = \bigcup_{i\in I} U_i$ and $V = \bigcup_{j\in J} U_j$ are two general open sets, with U_i, U_j semiprincipal open sets, and we suppose moreover $U \subseteq V$, then by definition V is equal to $W = \bigcup_{k\in I\cup J} U_k$, and so the canonical map $\Gamma(W) \to \Gamma(V)$ (recall that $\Gamma(V) = \varprojlim U_j$) is an isomorphism; but we also have a canonical map $\Gamma(W) \to \Gamma(U)$; we therefore obtain a canonical map $\Gamma(V) \to \Gamma(W)$. This we call the restriction map and we write it ρ_U^V. As usual, we have the functoriality (presheaf) conditions $\rho_U^U = \mathrm{id}_{\Gamma(U)}$ and, when $U \subseteq V \subseteq W$ then $\rho_U^W = \rho_U^V \rho_V^W$.

We call $\mathcal{U}_A$ the set of general open sets of $\operatorname{Spec} A$ (we are still not defining $\operatorname{Spec} A$). The relation $\subseteq$ is a (partial) order relation on $\mathcal{U}_A$. The smallest element is $\varnothing$ and the largest is $\operatorname{Spec} A$. Union is defined as follows: if $U_i = \bigcup_{k \in K_i} U_{ik}$ are general open sets, with U_{ik} semiprincipal open sets, then we let $\bigcup_{i \in I} U_i = \bigcup_{i \in I, k \in K_i} U_{ik}$. A routine check shows that this is well defined, and that it is in fact the smallest general open set containing all the U_i. When $I = \{1, \dots n\}$ is finite, we write $U_1 \cup \cdots \cup U_n$ for $\bigcup_{i=1}^n U_i$. Finally, intersection is defined as follows: if $U = \bigcup_{i \in I} U_i$ and $V = \bigcup_{j \in J} V_j$, then we let $U \cap V = \bigcup_{(i,j) \in I \times J} (U_i \cap V_j)$. Again, a routine check shows that this is well defined (check this first with V a semiprincipal open set, using $\varphi: A \to \Gamma(V)$ in the definition of equality of two general open sets), and that it is in fact the largest general open set contained in U and V. Of course, intersection and union are associative and commutative, the empty and full sets play the expected roles, intersection is distributive over union, and union is distributive over intersection whenever this makes sense.

Pointless spaces, skew ringed spaces and skew schemes.

By a pointless space we mean a partially ordered set $\mathcal{U}$ (whose order we will write $\subseteq$) with a smallest element $\varnothing$ and a largest element X (we then say that $\mathcal{U}$ is a pointless space on X; in fact, we sometimes say that X is a pointless space) such that any family (U_i) of elements of $\mathcal{U}$ has a least upper bound $\bigcup_i U_i$ any finite family has a greatest lower bound (we write $U \cap V$ for the greatest lower bound of U and V), and such that $\cap$ is distributive over $\cup$, which implies that $\cup$ is distributive over $\cap$ when it makes sense. When $\mathcal{U}$ is a pointless space on X, its elements are called open sets of X. We will use the terms "empty set", "full set", "union", "intersection", and so on, as seems natural, with no further comment. If U is an open set of a pointless space X, and V_i a family of open sets contained in U, then we say that the V_i cover U iff $U = \bigcup V_i$.

If $(X, \mathcal{U})$ is a pointless space (meaning that $\mathcal{U}$ is a pointless space on X), a presheaf (of rings) $\mathcal{F}$ is an object which to every open set U associates a ring $\Gamma(U, \mathcal{F})$ (or $\mathcal{F}(U)$) and to every inclusion $U \subseteq V$ of open sets associates a "restriction" homomorphism $\rho_U^V: \Gamma(V, \mathcal{F}) \to \Gamma(U, \mathcal{F})$, in such a way that

(o) $\Gamma(\varnothing, \mathcal{F})$ is the zero ring.

(i) For every $U \in \mathcal{U}$ we have $\rho_U^U = \mathrm{id}_{\Gamma(U, \mathcal{F})}$.

(ii) If $U \subseteq V \subseteq W$ then $\rho_U^W = \rho_U^V \rho_V^W$.

Moreover, $\mathcal{F}$ is said to be a sheaf iff it satisfies the following additionnal requirement:

(iii) If $V_i \subseteq U$ for every i, and the V_i cover U, then the canonical map $\hat{\rho}: \Gamma(U, \mathcal{F}) \to \varprojlim \Gamma(V_i, \mathcal{F})$ obtained from the restriction maps is an isomorphism. Here, $\varprojlim \Gamma(V_i, \mathcal{F})$ is the ring of "matching" families, i.e. families (s_i) such that $\rho_{V_i \cap V_j}^{V_i}(s_i) = \rho_{V_i \cap V_j}^{V_j}(s_j)$ for every i, j.

A pair $(\mathcal{U}, \mathcal{O})$, where $\mathcal{U}$ is a pointless space and $\mathcal{O}$ a sheaf on it, is called a skew ringed space (the term "skew" is used to emphasize on the possible noncommutativity). We will frequently say that X is a skew ringed space when what we really mean is that $(\mathcal{U}, \mathcal{O})$ is a skew ringed space with X the maximal element of $\mathcal{U}$. The sheaf $\mathcal{O}$ is called the structure sheaf of the skew ringed space.

If X, X' are skew ringed spaces with underlying pointless spaces $\mathcal{U}, \mathcal{U}'$, and structure sheafs $\mathcal{O}, \mathcal{O}'$, then a morphism $f: X \to X'$ is a pair $(f^\natural, f^\sharp)$, where $f^\natural: \mathcal{U}' \to \mathcal{U}$ is a map preserving both $\cap$ and $\cup$ (and therefore also preserving $\subseteq$), and $f^\sharp$ is a family of ring homomorphisms $f_U^\sharp: \Gamma(U, \mathcal{O}') \to$

$\Gamma(f^\natural(U),\mathcal{O})$, one for each $U\in\mathcal{U}'$, satisfying the "naturality" condition that, if $U\subseteq V$ then

$$\rho^{f^\natural(V)}_{f^\natural(U)} f^\sharp_V = f^\sharp_U \rho^V_U$$

(the first ρ is restriction for $\mathcal{O}$, and the second is restriction for $\mathcal{O}'$; note that $f^\natural(U)\subseteq f^\natural(V)$). Morphisms of skew ringed spaces are composed in the obvious way ($(\psi\varphi)^\natural = \varphi^\natural\psi^\natural$ and similarily for $(\psi\varphi)^\sharp$). Skew ringed spaces form a category; by an isomorphism of skew ringed spaces we shall naturally mean an isomorphism in that category.

If $(\mathcal{U},\mathcal{O})$ is a skew ringed space on X, and U is an open set (i.e. an element of $\mathcal{U}$), then we have a pointless space $\mathcal{U}|_U$ consisting of those elements of $\mathcal{U}$ which are included in U, and a sheaf $\mathcal{O}|_U$ on it, which is just equal to $\mathcal{O}$ on elements of $\mathcal{U}|_U$. The pair $(\mathcal{U}|_U,\mathcal{O}|_U)$ forms a skew ringed space, which we call an open subspace of X. We will of course write U for this space. We have a morphism of ringed spaces $i\colon U\to X$, called the inclusion map, defined as follows: $i^\natural$ takes an element V of $\mathcal{U}$ to its intersection with U, and $i^\sharp_V$ takes an element $s\in\Gamma(V,\mathcal{O})$ to its restriction to $V\cap U$, considered as an element of $\Gamma(i^\natural(V),\mathcal{O}|_U)$.

Now, if A is a ring, we have defined two objects: a set $\mathcal{U}_A$ of "general open sets", and a "structure sheaf" $\mathcal{O}_A$. Now $\mathcal{U}_A$ is a pointless space, and it is easy to see that $\mathcal{O}_A$ is indeed a sheaf on it (remarkably, it is easier to see than in the commutative case — of course, there is no miracle, and that is just due to the way we have defined open sets and coverings). So $(\mathcal{U}_A,\mathcal{O}_A)$ is a skew ringed space, which we write $\operatorname{Spec} A$ and call the spectrum of A.

If $\varphi\colon A\to B$ is a ring homomorphisms, then we define a morphism $\varphi^*\colon\operatorname{Spec} B\to\operatorname{Spec} A$ as follows. If $f\in A$ then we let $(\varphi^*)^\natural(D(f_1;\ldots;f_n)) = D(\varphi(f_1);\ldots;\varphi(f_n))$, and we let

$$(\varphi^*)^\sharp_{D(f_1;\ldots;f_n)}\colon A_{f_1;\ldots;f_n}\to B_{\varphi(f_1);\ldots;\varphi(f_n)}$$

be the multicalization of φ. Clearly, $(\varphi^*)^\natural$, thus defined on semiprincipal open sets, preserves intersection; we define it on general open set by imposing that it preserve union: the very definition of equality of two general opens sets ensures that this is well defined. Finally, for $U=\bigcup U_i$ with the U_i semiprincipal open sets, we define $(\varphi^*)^\sharp_U$ as the projective limit of the $(\varphi^*)^\sharp_{U_i}$. It is then a simple exercice to verify that everything fits together nicely and that $((f^*)^\natural,(f^*)^\sharp)$ is a morphism of skew ringed spaces. Note that not all morphisms of skew ringed spaces from $\operatorname{Spec} B$ to $\operatorname{Spec} A$ are not of this form; however, we do have the following pleasant result:

Proposition. *If $\lambda\colon\operatorname{Spec} B\to\operatorname{Spec} A$ is an isomorphism of skew ringed spaces, then there exists an isomorphism $\varphi\colon A\to B$ such that $\lambda=\varphi^*$.*

Proof: Since $\lambda^\natural$ is an isomorphism of partially ordered sets, it is takes $\operatorname{Spec} A$ to $\operatorname{Spec} B$. Therefore we have an isomorphism $\lambda^\natural_{\operatorname{Spec} A}\colon A\to B$. Composing with the isomorphism $\operatorname{Spec} A\to\operatorname{Spec} B$ induced by the inverse of that isomorphism, we can assume that $B=A$ and that $\lambda^\natural_{\operatorname{Spec} A}\colon A\to A$ is the identity, and we must prove that λ is the identity.