(0)Motivation. *There are several hints that the category of rings*[*] *is inadequate for doing noncommutative algebraic geometry (I refer to the reader to my other texts on the subject for a discussion of them). Essentially, the problem is this. If* R *is a commutative ring and* f *an element of* R*, and we want to construct a ring which is "just like* R*" except that* f *is zero, we take* $R/(f)$ *(the quotient of* R *by the ideal generated by* f*); geometrically, that corresponds to restricting to the locus of zeroes of* f*, and the quotient thus obtained is zero if and only if* f *is invertible in* R*, which is reasonable because saying that* f *is invertible means it has no zeroes, in other words that its locus of zeroes is empty. Now if* R *is a ring that is not necessarily commutative and we want to do the same thing, we might be tempted to consider* $R/(f)$*, where* (f) *is the two-sided ideal generated by* f *(because one can only quotient rings by two-sided ideals). Unfortunately, the condition for* $R/(f)$ *to be zero is a rather uninteresting condition, namely that there exist* a_i, b_i *such that* $\sum a_i f b_i = 1$*. One would much prefer this condition to be that* f *be invertible. This is one way of saying that a general ring* R *has too few ideals: the ideal generated by* f *gets too big too easily. Another point is that the ring* $\mathbb{M}_n(k)$ *of* $n \times n$ *matrices over a field* k *is simple, which is somewhat a contradiction with its geometrical content (which I have previously argued is something like "*n *points moving around"). Now if one is to construct more ideals, in other words, define something weaker than two-sided ideals, let us call them "weak ideals", one also has to allow quotienting by these generalized ideals, and thus to extend the category of rings to something larger which I mockingly call "nibelungen" as they are something of an impossible quest. In more categorical terms, the category of rings has satisfying projective limits (products and equalizers) but its inductive limits, and more precisely its coequalizers, are not good. So we want to (fully) embed the category of rings in a larger category, that of nibelungen, in such a way that the inclusion functor have a left adjoint (so as to guarantee that it will commute with projective limits), the "ring of the nibelungen" functor.*

(1)Proposition. *Let* R *be a ring and* I *a subset of* R*. Then the following conditions are equivalent:*

(i) If $a_1, \ldots, a_k, a'_1, \ldots, a'_k \in R$ *and* $f_1, \ldots, f_k \in I$ *are such that*

$$a_1 f_1 + \cdots + a_k f_k = f_1 a'_1 + \cdots + f_k a'_k \tag{†}$$

then this element belongs to I*.*

(ii) I *is the intersection of the left ideal and the right ideal it generates.*

(iii) I *is the intersection of the left ideal and the right ideal generated by some subset* S *of* R*.*

(iv) I *is the intersection of some left ideal and some right ideal.*

Proof: (ii)$\Longrightarrow$(iii)$\Longrightarrow$(iv)$\Longrightarrow$(i) are obvious. Now let us prove (i)$\Longrightarrow$(ii). Call $\mathfrak{L}$ (resp.$\mathfrak{R}$) the left (resp.right) ideal generated by I. We have $I \subseteq \mathfrak{L} \cap \mathfrak{R}$ and it remains to prove that $\mathfrak{L} \cap \mathfrak{R} \subseteq I$. Now an element of $\mathfrak{L}$ can be written in the form $a_1 f_1 + \cdots + a_k f_k$ for some $a_1, \ldots, a_k \in R$ and $f_1, \ldots, f_k \in I$. Similarly,

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[*] *By a "ring" we mean a not necessarily commutative ring with a unit element; morphisms of rings are supposed to take the unit element to the unit element.*

an element of $\mathfrak{R}$ can be written in the form $g_1a'_1 + \cdots + g_\ell a'_\ell$ with $b_1, \ldots, b_\ell \in R$ and $g_1, \ldots, g_\ell \in I$. Now for an element of the intersection, by reordering and introducing zeroes as necessary we can assume that $\ell = k$ and that the f_i and the g_i coincide, so that our element is written in the form (†) and (i) tells us that it belongs to I, which was what we wanted.

(2)Definition. *A subset I of a ring R that satisfies the equivalent conditions just cited will be called a* <u>weak ideal</u> *of R.*

(3)Proposition. *Any intersection of weak ideals is again a weak ideal. Consequently, for any $S \subseteq R$, there exists a smallest weak ideal containing S (namely, the intersection of all weak ideals containing S). It will be called the weak ideal* <u>generated</u> *by S. It coincides with the intersection of the left and right ideals generated by S, and also with the set of all expressions of the form (†) where $a_1, \ldots, a_k, a'_1, \ldots, a'_k \in R$ and $f_1, \ldots, f_k \in S$.*

Proof: Trivial.

(4)Corollary. *An element $f \in R$ generates the unit weak ideal iff it is invertible (that is, left and right invertible).*

(5)Definition. *Let I be a weak ideal. Then the left (resp.right) ideal it generates is called the* <u>left</u> *(resp.*<u>right</u>*)* <u>friend</u> *of I. The pair $(\mathfrak{L}, \mathfrak{R})$, where $\mathfrak{L}$ (resp.$\mathfrak{R}$) is the left (resp.right) friend of I, is called the* <u>friend pair</u> *of I. The two-sided ideal generated by I, which is also the two-sided generated by any of its friends, is called the* <u>cousin</u> *of I.*

(6)Remark. *Any left ideal is the left friend of some weak ideal, simply because any left ideal is itself a weak ideal. Similarily for right ideals. However, any pair $(\mathfrak{L}, \mathfrak{R})$, where $\mathfrak{L}$ (resp.$\mathfrak{R}$) is a left (resp.right) ideal, is not necessarily the friend pair of a weak ideal (consider the commutative case for an obvious counterexample, as two-sided, left, right and weak ideals all coincide); if it is, however, then the weak ideal in question is $\mathfrak{L} \cap \mathfrak{R}$.*

(7)Definition. *A* <u>nibelung</u> *is a pair (R, I), where R is a ring and I a weak ideal of R. The abelian group R/I is called the* <u>underlying additive structure</u> *to the nibelung (R, I). We will often write R/I when talking about the nibelung itself. If $\mathfrak{A}$ is the cousin of I, the ring $R/\mathfrak{A}$ will be called the* <u>ring of the nibelung</u> *R/I.*

(8)Discussion. *We now wish to define morphisms of nibelungen. This presents some difficulty so we should proceed slowly. First, the category of rings should be a full subcategory of the category of nibelungen, i.e.morphisms from a ring R to a ring S should be the same whether we look at them in the category of rings or in that of nibelungen. Of course, by a ring R as a nibelung we mean any quotient R'/I where I is a two-sided ideal in R' with $R'/I \cong R$ as rings (we should arrange for all these nibelungen to be isomorphic). Morphisms from a nibelung to a ring also present no difficulty: a morphism from R/I to S (with R and S rings and I a weak ideal of R) should be the same as a morphism from R to S which is zero on I; in this case, it is also necessarily zero on the cousin of I; in other words, it is the same as a morphism from the ring of the nibelung R/I to S. This is exactly what we wanted since we wanted the "ring of the nibelungen" functor to be the left adjoint to the inclusion functor of rings in nibelungen.*

(9)Discussion, continued. *Now morphisms from rings* to *nibelungen are much more delicate. The simplest case is when the ring is $\mathbb{Z}$ (or the base field k if we are working over a base field). The ring $\mathbb{Z}$ is the initial object in the category of rings and we have no reason to want to change that in the category of nibelungen (although it is an inductive limit), so let $\mathrm{Hom}(\mathbb{Z}, S/J)$ consist of a single element as usual. Now consider a slightly more complicated ring, namelt $\mathbb{Z}[x] = \mathbb{Z}\langle x\rangle$ (here the angle brackets mean that the variables are not supposed to commute; of course when there is only one it doesn't make any difference). In the commutative case, a morphism from $\mathbb{Z}[x]$ to S is the same as an element of S (in sophisticated terms, the ring $\mathbb{Z}[x]$ represents the "underlying set" functor). This is also true for noncommutative rings: giving the image of x determines that of $x^2, x^3, \ldots$ and thus the image of any element of $\mathbb{Z}[x]$. So should we say that a morphism from $\mathbb{Z}[x]$ to a nibelung S/J is just an element of (the abelian group) S/J? That may sound reasonable, but then we have no way of calculating the image of x^2 (because there is no multiplication operation on S/J). Of course, one does not need morphisms to be actual maps so who cares about the image of x^2? But actually, one does, because by composing with the morphism $\mathbb{Z}[y] \to \mathbb{Z}[x]$ that takes y to x^2 one should find the image of x^2. So this first attempt fails. Now I suggest defining a morphism from $\mathbb{Z}[x]$ to S/J as an element a of S, where two such morphisms $x \mapsto a$ and $x \mapsto b$ are deemed equal, not (as previously) when $a - b \in J$ but rather when $a^n - b^n \in J$ for all $n \geq 0$. This way, the image by the morphism of any element of $\mathbb{Z}[x]$ in S/J is well-defined, and in fact these determine the morphism, so that we can view it as a particular kind of map from $\mathbb{Z}[x]$ to (the set) S/J.*

(10)Discussion, continued. *It then doesn't take much imagination to figure out what a morphism from $\mathbb{Z}\langle x_1, \ldots, x_r\rangle$ to S/J will be, or even from $\mathbb{Z}\langle x_i\rangle_{i\in\Lambda}$ to S/J (where $(x_i)_{i\in\Lambda}$ is an arbitrary family of noncommuting variables): it is defined by giving a family $(a_i)_{i\in\Lambda}$ of elements of S, where we identify two families (a_i) and (b_i) when for any $i_0, \ldots, i_s \in \Lambda$ we have $a_{i_0} \cdots a_{i_s} - b_{i_0} \cdots b_{i_s} \in J$. In other words, a morphism from $\mathbb{Z}\langle x_i\rangle_{i\in\Lambda}$ to S/J is a map of sets (necessarily a morphism of abelian groups in fact) that lifts to a morphism of rings from $\mathbb{Z}\langle x_i\rangle_{i\in\Lambda}$ to S. This lifting is not unique in general, and we only demand its existence — it is not part of the data defining the morphism.*

(11)Discussion, end. *So what about morphisms from an arbitrary ring R to a nibelung S/J? According to what we have just said, we might be tempted to define them as maps from R to S/J for which there exists a lifting to a morphism of rings from R to S. Now that is not a good definition, because even in the case where S/J is a ring (that is, J is a two-sided ideal) it does not give all the usual morphisms: a morphism of rings $R \to S/J$ does not always lift to a morphism of rings $R \to S$ whereas it does in the case $R = \mathbb{Z}\langle x_i\rangle_{i\in\Lambda}$ (we say that $\mathbb{Z}\langle x_i\rangle_{i\in\Lambda}$ is a projective object in the category of rings). So to define morphisms from R to S/J we first write R in the form R_0/K where R_0 is a free ring such as previously, say $R_0 = \mathbb{Z}\langle x_i\rangle_{i\in R}$, and I is a two-sided ideal of relations. And then a morphism from R to S/J is but a morphism from R_0 to S/J that factors through K (in other words, that is zero on I: zero is a well-defined element of S/J). Getting morphisms from between arbitrary nibelungen R/I and S/J is then straightforward: just demand that the morphism factor through K and I. For those who did not follow the discussion, we now formulate the actual definition:*

(12)Definition. *Let R/I and S/J be nibelungen. Then a morphism $\varphi\colon R/I \to S/J$ is a map of the underlying sets (which is also, necessarily, a morphism of the underlying abelian groups) such that there exists a ring R_0, a morphism of rings $\tilde{\varphi}\colon R_0 \to S$ and a surjective morphism of rings $\psi\colon R_0 \to R$ such that for $x \in R_0$ the image by φ of the class of $\psi(x)$ in R/I is the same as the class of $\tilde{\varphi}(x)$ in S/J, that is $\overline{\tilde{\varphi}(x)} = \varphi(\overline{\theta(x)})$. In other words, a morphism of nibelungen is a map of the underlying sets that lifts in some way to a morphism of rings.*

(13)Proposition. *We have the following properties:*

- *(o) In the previous definition, we can take for R_0 a free ring, that is, a ring of the form $\mathbb{Z}\langle x_i\rangle_{i\in\Lambda}$ where $(x_i)_{i\in\Lambda}$ is a family of noncommuting coordinates.*
- *(oo) In the previous definition, we can replace S by any ring S_0 of which S is a quotient: it is equivalent to demand for the morphism to lift to S_0 as to demand for it to lift to S.*
- *(i) If R/I is a nibelung, then the identity map $1_{R/I}\colon R/I \to R/I$ is a morphism of nibelungen.*
- *(ii) If $\varphi\colon R/I \to S/J$ and $\psi\colon S/J \to T/K$ are morphisms of nibelungen then $\psi\varphi\colon R/I \to T/K$ is a morphism of nibelungen.*

Consequently, nibelungen form a category, and when we speak of isomorphism of nibelungen we mean an isomorphism in that category.

- *(iii) Rings for a full subcategory of the category of nibelungen (where we identify the ring R with the nibelung $R/0$).*
- *(iv) If R is a ring and I a two-sided ideal of R then the nibelung R/I is isomorphic (as a nibelung) to the ring R/I.*

Proof: (o): Any ring is a quotient of a free ring, so let R_0 be a quotient of a free ring R_1, and extend the morphism $R_0 \to S$ to R_1 in the obvious (and unique) way.

(oo): Taking R_0 to be a free ring, as we can by property(o) above, the morphism $R_0 \to S$ can be lifted to a morphism $R_0 \to S_0$ by a property of free rings (projectiveness).

(i): $1_{R/I}$ lifts to $1_R\colon R \to R$ which is a morphism of rings.

(ii): Let $\tilde{\psi}\colon S_0 \to T$ witness the fact that ψ is a morphism, and let $\tilde{\varphi}\colon R_0 \to S_0$ witness the fact that φ is one (we can take $\tilde{\varphi}$ with target S_0 by property(oo) above). Then $\tilde{\psi}\tilde{\varphi}\colon R_0 \to T$ shows that $\psi\varphi$ is a morphism of nibelungen.

(iii): Suppose R and S are rings and $\varphi\colon R \to S$ is a morphism of nibelungen. In other words, there exists a surjective morphism of rings $\theta\colon R_0 \to R$ and a morphism or rings $\tilde{\varphi}\colon R_0 \to S$ such that $\tilde{\varphi} = \varphi\theta$. Now if $x, y \in R$ and $\dot{x}, \dot{y} \in R_0$ represent them then $\dot{x}\dot{y}$ represents xy and we have $\varphi(xy) = \tilde{\varphi}(\dot{x}\dot{y}) = \tilde{\varphi}(\dot{x})\tilde{\varphi}(\dot{y}) = \varphi(x)\varphi(y)$ so that φ is a morphism of rings.

(iv): To make notations clearer, we let $R' = R/I$ be the quotient ring. So we are to compare the nibelung R/I with the nibelung $R'/0$. We define morphisms in both directions, namely the identities (both R/I and R' are the same abelian group). The identity map $R/I \to R'$ is a morphism of nibelungen because it obviously lifts to the canonical map $R \to R'$ which is a morphism of rings. The identity map $R' \to R/I$ is also a morphism of nibelungen because it lifts to the identity map $R \to R$.