A first introduction to
General Nonsense

Introduction: My purpose in writing this paper is first of all to test my own understanding of the theory of categories. There is nothing difficult about this theory; it is just very abstract. Indeed, it might be called pure abstraction, since it applies to a spectacularily wide range of parts of mathematics. All the general arguments are completely *trivial*, but that does not make them easy to produce, even less to follow, for when one starts talking about the universal object in a category whose objects are group objects in another category consisting of natural maps between functors of a third category, which might itself be a slice category, then intuition has a tendency to loose its footing. Much less trivial, however, are the arguments which apply to a particular category; indeed, though the general definitions are very simple, since they are designed to have a certain meaning in one category, they may be very difficult to understand in another category. For example, what in the world is an epimorphism in the category **HomoTop** of topological spaces with homotopy classes as morphisms? I do not know. And the same applies to many a question of that type. Therefore, while my goal has been to give as many examples as possible, by sometimes seeking to apply a strange concept to a category with which it had *a priori* nothing to do, I have been far from succeeding in being as systematic as I would have wished. Nevertheless, I believe the reading of this introduction might be a good exercice for the beginner who wishes to make sure he understood correctly all the basic concepts, and does not have a mental image biased by the too-intesive study of only one type of examples. Any extra illuminating (or, better even, confusing) examples or suggestions are welcome.

The prerequisites for reading this text are very basic. No prior knowledge of category theory is required, and only a very limited amount of algebra and topology is needed. On a few occasions, a mention of more sophisticated concepts (the amalgam of two groups, homotopy and singular homology, schemes, and various theorems in differential geometry) has been made, but it can always be skipped without otherwise impairing the comprehension.

1. Categories and morphisms.

A <u>category</u> $\mathbf{C}$ is a class of "objects" $\mathrm{ob}(\mathbf{C})$ together, with, for each objects $A, B \in \mathrm{ob}(\mathbf{C})$, a set $\mathrm{Hom}(A, B)$ of "morphisms" from A to B, and a composition law

$$\mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C)$$

for all $A, B, C \in \mathrm{ob}(\mathbf{C})$, which is associative whenever defined, and admits an identity element $1_A \in \mathrm{Hom}(A, A)$ for each $A \in \mathrm{ob}(\mathbf{C})$.

Whenever $\alpha \in \mathrm{Hom}(A, B)$ is a morphism, A is called its <u>domain</u> (or <u>source</u>) and will be written $\mathrm{dom}\,\alpha$, and B its <u>range</u> (or <u>target</u>, or <u>goal</u>, or <u>codomain</u>), and will be written $\mathrm{ran}\,\alpha$. To say that α is a morphism with domain A and range B, one also writes $\alpha\colon A \to B$.

Thus, the definition of a category states that the composition $\beta\alpha$ of two morphisms is defined iff $\mathrm{ran}\,\alpha = \mathrm{dom}\,\beta$, in which case it is a morphism with domain $\mathrm{dom}\,\alpha$ and range $\mathrm{ran}\,\beta$. Whenever α, β, γ are morphisms such that $\mathrm{ran}\,\alpha = \mathrm{dom}\,\beta$ and $\mathrm{ran}\,\beta = \mathrm{dom}\,\gamma$, in which case the two morphisms $\gamma(\beta\alpha)$ and $(\gamma\beta)\alpha$ are defined, they are equal. Lastly, for each object A in $\mathbf{C}$ there exists a (necessarily unique) morphism 1_A with domain and range both A, and is such that $1_A\alpha = \alpha$ for any morphism with range A, and $\alpha 1_A = \alpha$ for any morphism with domain A.

> The category **Set** is the category whose objects are all sets, and whose morphisms are just ordinary maps between sets; in other words, if A and B are sets, then $\mathrm{Hom}(A, B)$ is the set of all functions from A to B. Composition is defined in the usual way, and 1_A is the identity on A for each set A.
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> The category **PSet** is the category whose objects are all pairs (A, x) with A a (non empty) set and $x \in A$ (the "base point"), and whose morphisms are maps of sets which preserve base points; in other words, if (A, x) and (B, y) are pointed sets, then $\mathrm{Hom}((A, x), (B, y))$ is the set of all functions α from A to B such that $\alpha(x) = y$, with composition defined in the usual way.
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> The category **Top** is the category whose objects are all topological spaces, and whose morphisms are continuous maps between them, with composition defined in the usual way. The category **HausTop** is the category whose objects are Hausdorff topological spaces, and whose

morphisms are continuous functions between them. The reader will have no difficulty guessing what the categories **PTop** (pointed topological spaces) and **PHausTop** (pointed Hausdorff spaces) are.

The category **Group** is the category whose objects are all groups, and whose morphisms are group morphisms in the usual sense of the word. The category **AbGroup** is the category whose objects are abelian groups, and whose morphisms are group morphisms.

The category **POSet** is the category whose objects are partially ordered sets, and whose morphisms are order-preserving functions between them; in other words, $\alpha \in \mathrm{Hom}(A, B)$ with A, B two partially ordered sets, whenever $x \leq y$ implies $\alpha(x) \leq \alpha(y)$ for all $x, y \in A$.

If P is a partially ordered set, then it can be viewed as a category in the following way: let the objects be the elements of P, and let $\mathrm{Hom}(u, v)$ consist of just one element whenever $u \leq v$, and be empty otherwise. The morphisms in this category are not viewed as functions. Also note that the set $\mathrm{Hom}(u, v)$ has at most one element for any two objects u, v. This makes composition of morphisms obvious to define: if $u \leq v \leq w$, then there is only one morphism in $\mathrm{Hom}(u, v)$ and only one in $\mathrm{Hom}(v, w)$, and their composite is the unique element of $\mathrm{Hom}(u, w)$. The "identity" on u, is, of course, the unique morphism in $\mathrm{Hom}(u, u)$. Note that this whole construction also works when P is a preordered set (i.e. a set with a reflexive and transitive binary relation on it.

The category **Ring** is the category whose objects are (unitary) rings and whose morphisms are ring homomorphisms (sending the unit element of the domain to the unit element of the range). The category **ComRing** is the category whose objects are commutative rings and whose morphisms are ring homomorphisms. The category **Field** is the category whose objects are fields and whose morphisms are field homomorphisms.

If G is a group, then it can be made into a category in the following way: take only one object, say O, and let $\mathrm{Hom}(O, O) = G$ with the composition defined as the group law on G. Naturally, 1_O is the unit element on G.

If G is a group, let G**Set** be the category whose objects are G-sets (that is, sets on which G acts), and whose morphisms are maps which respect the action of G, in other words, $\alpha \in \mathrm{Hom}(A, B)$ (where A, B are two objects) iff $\alpha(g \cdot x) = g \cdot \alpha(x)$ for each $x \in A$.

If R is a ring, let R**Mod** be the category whose objects are R-modules, and whose morphisms are linear maps of R-modules. Moreover, let R**Alg** be the category whose objects are R-algebras, and whose morphisms are homomorphisms of R-algebras.

Recall that two continuous functions $f, g: X \to Y$ between two topological spaces are said to be <u>homotopic</u> iff there exists $\Theta: [0, 1] \times X \to Y$ such that $f = \Theta(0, \cdot)$ and $g = \Theta(1, \cdot)$. This is an equivalence relation on the set of continuous functions from X to Y, whose equivalence classes are called <u>homotopy classes</u>. The composite of two homotopy classes can be defined, for, if f' is homotopic to f and g' to g, then gf is homotopic to $g'f'$. The category **HomoTop** is the category whose objects are topological spaces and whose morphisms are homotopy classes, with the composition defined as described above, and with 1_A being the homotopy class of the identity for any topological space A. The categories **HausHomoTop**, **PHomoTop** and **PHausHomoTop** are defined in the obvious way.

The same sort of ideas applies in homological algebra: let **GradGroup** be the category whose objects are just $\mathbb{Z}$-sequences of groups, and whose morphisms are $\mathbb{Z}$-sequences of group homomorphisms; let **GradAbGroup** be the category whose objects are $\mathbb{Z}$-sequences of abelian groups, and whose morphisms are $\mathbb{Z}$-sequences of abelian group homomorphisms. Let ∂**AbGroup** be the category whose objects are complexes of abelian groups, that is, $\mathbb{Z}$-sequences (K_n) of abelian groups together with "boundary" homomorphisms $\partial_n: K_n \to K_{n-1}$ such that $\partial_n \partial_{n+1} = 0$ for each integer n, and whose morphisms are chain maps, that is, $\mathbb{Z}$-sequences of abelian group homomorphisms $f_n: K_n \to K'_n$ such that $\partial'_n f_n = f_{n-1} \partial_n$ for each n. We say that two chain maps $f, g: K \to K'$ are <u>homotopic</u> iff there exists a $\mathbb{Z}$-sequence of abelian group homomorphisms $s_n: K_n \to K'_n$ such that $f_n - g_n = \partial'_{n+1} s_n + s_{n-1} \partial_n$ for each $n \in \mathbb{Z}$. After verifying that this is an equivalence relation compatible with composition, we let **Homo∂AbGroup** be the category whose objects are complexes of abelian groups and whose morphisms are homotopy classes of chain maps between them.

For $0 \leq r \leq \omega$ (this means $r \in \mathbb{N}$ or $r = \infty$ or $r = \omega$) the category C^r**Man** is the category of (paracompact separable) C^r-manifolds, whose morphisms are C^r maps. (C^ω means "real analytic"). One can also define the category **HolMan** whose objects are complex analytic (i.e. holomorphic) manifolds and whose morphisms are holomorphic maps between them.

There is a category **Scheme**, whose objects are schemes and whose morphisms are regular maps.

If $\mathbf{C}$ is a category, and J any object of $\mathbf{C}$, then there is a category $\mathbf{C} \downarrow J$ called the slice category whose objects are "objects over J", that is, $\mathbf{C}$-morphisms $A \to J$, and whose morphisms are $\mathbf{C}$-morphisms which make the obvious triangle commute. In other words, a morphism $\alpha \in \mathrm{Hom}(f, g)$ with $f: A \to J$ and $g: B \to J$ is a $\mathbf{C}$-morphism $\alpha: A \to B$ such that $f = g\alpha$ (in $\mathbf{C}$). Composition of morphisms in $\mathbf{C} \downarrow J$ is defined in the obvious way, and the identity morphism on $f: A \to J$ is just 1_A (it satisfies $f = f1_A$). J itself can be considered as an object of $\mathbf{C} \downarrow J$, viz. $1_J: J \to J$.

If $\mathbf{C}$ is a category, then there is a category $\mathbf{C^{op}}$ whose objects are the same as those of $\mathbf{C}$, but all of whose arrows are reversed. This means that

$$\mathrm{Hom}_{\mathbf{C^{op}}}(A, B) = \mathrm{Hom}_{\mathbf{C}}(B, A)$$

The composition of morphisms is defined in the following way: if $\alpha: A \to B$ and $\beta: B \to C$ are morphisms in $\mathbf{C^{op}}$ (which means that, in $\mathbf{C}$, $\alpha: B \to A$ and $\beta: C \to B$), then $\beta\alpha$ in $\mathbf{C^{op}}$ is defined to be $\alpha\beta$ in $\mathbf{C}$, which makes sense because $\alpha\beta: C \to A$ in $\mathbf{C}$ so $\beta\alpha: A \to C$ in $\mathbf{C^{op}}$. Of course, 1_A in $\mathbf{C^{op}}$ is just 1_A in $\mathbf{C}$.

If $\mathbf{C}$ and $\mathbf{D}$ are two categories, then there is a category $\mathbf{D^C}$ whose objects are functors (see below) from $\mathbf{C}$ to $\mathbf{D}$ and whose morphisms are natural maps (see below) between functors).

In a category $\mathbf{C}$, a morphism $\alpha: A \to B$ is said to be an isomorphism iff there exists $\beta: B \to A$ such that $\beta\alpha = 1_A$ and $\alpha\beta = 1_B$. Two objects are said to be isomorphic iff there exists an isomorphism between them.

A morphism $\alpha: A \to B$ is said to be mono (or a monomorphism) iff it is left cancellable, i.e. iff $\alpha f = \alpha g$ implies $f = g$ for any two morphisms f, g with range A. A morphism $\alpha: A \to B$ is said to be epi (or an epimorphism) iff it is right cancellable, i.e. iff $f\alpha = g\alpha$ implies $f = g$ for any two morphisms f, g with domain B. Note that "mono&epi does not imply iso": there may exist a morphism with is both mono and epi between two objects in a category, without these two objects being isomorphic.

Let us (temporarily) say that two monomorphisms $\alpha: A \to C$ and $\beta: B \to C$ in a category are equivalent iff there exists an isomorphism $h: A \to B$ such that $\alpha = \beta h$. An equivalence class of monomorphisms under this equivalence relation is called a subobject of C. There is a dual notion of quotient object: let us say that two epimorphisms $\alpha: C \to A$ and $\beta: C \to B$ in a category are equivalent iff there exists an isomorphism $h: A \to B$ such that $\beta = h\alpha$. An equivalence class of epimorphisms under this equivalence relation is called a quotient object of C.

It is well known that, in the category **Set**, an isomorphism is just a bijection, a monomorphism an injection (one-to-one map), and an epimorphism a surjection (onto map). [For example, let us prove the last statement. A surjection is clearly right cancellable. Conversely, if $\alpha: A \to B$ is right cancellable, and $y \in B$, then there consider the two maps $f, g: B \to \{0, 1\}$, where f takes any element to 0 and g takes any element other then y to 0 and takes y to 1. Obviously, $f \neq g$, so we must have $f\alpha \neq g\alpha$. This obviously ensures that α takes the value y. So α is onto.] A subobject of a set C is just a subset of C in the following sense: for any subset $A \subseteq C$, the inclusion map $\iota: A \to C$ is mono, and thus defines a subobject of C; in fact, a monomorphism $B \to C$ defines the same subobject iff its image is A. So we can identify subsets of C and subobjects of C. A similar argument shows that a subobject of an object C is just an equivalence relation on C (or, equivalently, a partition of C).

In the category **PSet** of pointed sets, an isomorphism is just a bijection which preserves the base point. A monomorphism (resp. epimorphism) is just an injective (resp. surjective) morphism. A subobject of a pointed set is a subset with the same base point, and a quotient object of a pointed set is a partition, with the equivalence class of the base point as base point.

In the category **Top**, isomorphisms are just homeomorphisms, monomorphisms are injective continuous maps, and epimorphisms are surjective continuous maps [apply the same reasoning as for sets, putting the coarse topology on $\{0, 1\}$]. A subobject of a space C is *not* just a subspace of C since a mono map need not be a homeomorphism onto its image (an embedding); rather, a subobject of C is a subset of C with a topology which is finer than the topology it inherits from C. Similarily, a quotient object of C is a quotient of the set C with a topology which is coarser than the ordinary quotient topology. In the case of **HausTop**, we encounter a difficulty: whereas it is true that isomorphisms are just homeomorphisms and that monomorphisms are injective continuous maps, it is *not* true that all epimorphisms are surjective. Indeed, epimorphisms are precisely those continuous maps whose image is dense. [Proof: Suppose $\alpha: A \to B$ is epi. Let D be its image, and F the closure of D. Consider the topological space C obtained by gluing along F two copies of B. Since F is closed, C is Hausdorff. Let $f, g: B \to C$ be the canonical maps to the first and second copy. Clearly, $f\alpha = g\alpha$, so $f = g$. Therefore, $F = B$, and the image of α is dense. For the converse, if the image of α is dense, and f, g are such that $f\alpha = g\alpha$, then f and g coincide on the image, and since their range is Hausdorff, everywhere on B, which means $f = g$. So α is epi, which completes the proof.]

In the category **Group**, an isomorphism is an ordinary group isomorphism, a monomorphism is an injective morphism. An epimorphism is indeed a surjective morphism, but that is not so trivial [Proof: Suppose $\alpha: A \to B$ is epi, and let H be its image. Now consider the amalgam C of B with itself along H, and let $f, g: B \to C$ be the canonical maps to the first and second copy of B. Clearly, $f\alpha = g\alpha$, so $f = g$. Therefore, $H = B$, and α is surjective. The converse is obvious. (Of course, the non trivial part of the proof is the existence and the description of the amalgam.)] A subobject of a group C is just a subgroup of C, and a quotient object is a quotient group. In the category **AbGroup**, the same results hold, with simpler proof: an isomorphism is a bijective morphism, a monomorphism is an injective morphism, and an epimorphism is a surjective morphism. [Proof of the latter: If $\alpha: A \to B$ is surjective, it is obviously epi. Now if α is epi, let H be its image. Let $g: B \to B/H$ be the canonical morphism, and $f: B \to B/H$ be the zero morphism. Obviously $f\alpha = g\alpha$ (both are the zero morphism), so $f = g$, which means exactly $H = B$.] A subobject of an abelian group C is a subgroup of C, and a quotient object is a quotient group.

In the category **POSet**, an isomorphism is what is normally meant by an isomorphism of partially ordered sets (i.e. an order-preserving bijection whose inverse is also order-preserving). A monomorphism is an injective order-preserving function, and an epimorphism is a surjective order-preserving function. [Proof of the latter: One way is obvious. To see the other, suppose $\alpha: A \to B$ is epi, and $y \in B$. Let $C = \{0, 1\}$ with the obvious order. Set $f: t \mapsto 1$ if $t \geq y$ and 0 otherwise; set $g: t \mapsto 1$ if $t > y$ and 0 otherwise. Then f and g are different and order-preserving. So $f\alpha$ and $g\alpha$ must also be distinct. Therefore, α takes the value y. So α is onto.] A subobject of a partially ordered set is a subset of that set, with a coarser order (for essentially the same reasons as in the case of **Top**); a quotient object is a quotient of the partially ordered set by an equivalence relation which is compatible with the order, together with an order finer than the quotient order.

If a partially ordered set is made into a category, its only isomorphisms are identity morphisms, whereas *all* morphisms are mono and epi. A subobject of u is an element less or equal to u, and a quotient object of u is an element greater or equal to u.

In the category **Ring**, isomorphisms are ordinary ring isomorphisms. Monomorphisms are injective morphisms [Proof: Suppose $\alpha: A \to B$ is mono where A, B are rings, and $x \in A$ is in the kernel. Consider the two morphisms $f, g: \mathbb{Z}[X] \to A$, given by $f(X) = 0$ and $g(X) = x$. Clearly, $\alpha f = \alpha g$ and so $f = g$, which means $x = 0$. So α is injective. The converse is obvious.] What epimorphisms are I cannot guess. In **ComRing**, iso and mono mean what they should. An epimorphism is not necessarily surjective, here is a spectacular example: the canonical injection $\mathbb{Z} \to \mathbb{Q}$ is epi (and mono, of course) [Proof: If f, g are homomorphisms of commutative rings with domain $\mathbb{Q}$ which agree on $\mathbb{Z}$ then for any $q \in \mathbb{N}^*$ we have $g(q)\, f(\frac{1}{q}) = f(q)\, f(\frac{1}{q}) = f(1) = 1$ so $f(\frac{1}{q})$ is the inverse of $g(q)$; but $g(\frac{1}{q})$ also, and so $f(\frac{1}{q}) = g(\frac{1}{q})$. And therefore $f = g$.] A possible criterion is: $\alpha: A \to B$ is epi iff the map $b \mapsto 1 \otimes b$ is onto $B \otimes_{\alpha(A)} B$.

In the category **Field**, *all* morphisms are mono (they are all injective). Isomorphisms, of course, are just what they should be. It is perhaps possible to give a reasonable explanation of what an epimorphism is, but this certainly involves Galois theory, and I gave up. At any rate, subobjects are subfields.

If G is a group, then in the category G**Set**, an isomorphism is a bijective morphism. A monomorphism is an injective morphism [Proof: Suppose $\alpha: A \to B$ is mono. For any $x, y \in A$, consider the morphisms $f, g: G \to A$ (where G acts upon itself by left multiplication) such that $f(1) = x$ and $g(1) = y$. If $\alpha(x) = \alpha(y)$ then $\alpha f = \alpha g$, so $f = g$ since α is mono, and therefore $x = y$. This means that α is injective. The converse is obvious.] And an epimorphism is a surjective morphism, for once that is easy to prove [consider the set $\{0, 1\}$ with the trivial action, and reason as for **Set**]. A subobject of a G-set is a sub-G-set (that is, a subset invariant under the action of G), and a quotient of a G-set is a quotient set by an equivalence relation which is compatible with the action, on which G acts in the obvious way.

If R is a ring, then in R**Mod**, isomorphisms are usual module isomorphisms, monomorphisms are injective morphisms, and epimorphisms are surjective morphisms. [Proof just like the case of abelian groups.]

Two topological sets are said to have the same homotopy type whenever they are isomorphic in the category **HomoTop**.

An isomorphism in C^r**Man** is a C^r-diffeomorphism.

In the slice category $\mathbf{C} \downarrow J$, an isomorphism from $A \to J$ to $B \to J$ is just a morphism (in $\mathbf{C} \downarrow J$) which happens to be an isomorphism in $\mathbf{C}$. A morphism (in $\mathbf{C} \downarrow J$) which is mono (resp. epi) in $\mathbf{C}$, is mono (resp. epi) in $\mathbf{C} \downarrow J$. The converse does not hold in general.

Passing to $\mathbf{C}^{\mathbf{op}}$ dualizes all notions, so monomorphisms become epi, and epimorphisms become mono. Subobjects give rise to quotient objects, and vice versa. Isomorphisms, of course, remain isomorphisms.

Finally, let us mention that an isomorphism in the category of functors $\mathbf{C}^{\mathbf{D}}$ is a natural isomorphism (see below).

A terminal (or universally attracting, or just universal) object in a category is an object 1 such that for each object A there exists a *unique* morphism $A \to 1$. An initial (or universally repelling, or couniversal) object in a category is an object 0 such that for each object A there exists a *unique* morphism $0 \to A$. It is clear that an initial or terminal object is unique up to isomorphism (this is in fact the starting point for the proof of the unicity of any universal or couniversal problem, see below).

Any morphism going *to* a universally repelling object is obviously epi, and any morphism coming *from* a universally attracting object is obviously mono.

In **Set**, there is an initial object, viz. $\varnothing$, and a terminal object, viz. any singleton. In **PSet**, a pointed singleton is both universally attracting and universally repelling.

In **Top**, there is an initial object, viz. $\varnothing$, and a terminal object, viz. any singleton. The same applies to **HausTop**. The reader will have no difficulty guessing what happens in **PTop** and **PHausTop**.

In the category **Group**, the group $\{1\}$ acts both as a universally attracting and univerrsally repelling object. The same applies to **AbGroup**.

In **POSet**, $\varnothing$ is initial and any singleton is terminal.

If a partially ordered set is viewed as a category, an initial object is a smallest element and a terminal object is a greatest element.

In the category **Ring**, there is an initial object, viz. $\mathbb{Z}$, and a terminal object, viz. $\{0\}$. The same applies to **CRing**. In **Field**, however, there is no initial nor terminal object (in the category $\mathbf{Field}_0$ of fields of caracteristic 0, there is an initial object, viz. $\mathbb{Q}$).

In the category **HomoTop**, there is an initial object, namely $\varnothing$. More important are the terminal objects: a terminal object in **HomoTop** is (by definition) a contractible space.

As for the category **Scheme**, it has a universal object, namely $\operatorname{Spec}\mathbb{Z}$. (Compare this with the fact that $\mathbb{Z}$ is initial in the category of commutative rings.) It also has an initial object, $\operatorname{Spec} 0$, but that fact is much less interesting (note, by the way, that $\operatorname{Spec} 0$ is *empty*, since the only ideal $\{0\}$ of 0 is *not* prime).

Now consider the slice category $\mathbf{C} \downarrow J$. If $\mathbf{C}$ has an initial object 0, then $0 \to J$ (the unique morphism from 0 to J) is an initial object in $\mathbf{C} \downarrow J$, as is readily checked. And the terminal object of $\mathbf{C} \downarrow J$ is just the identity morphism $1_J: J \to J$.

Of course, passing to $\mathbf{C}^{\mathbf{op}}$ turns initial objects into terminal objects, and vice versa.

If $(A_i)_{i\in I}$ is a family of objects of a category $\mathbf{C}$, a product of the family is an object P, together with a family $(\varpi_i)_{i\in I}$ of morphisms, $\varpi_i: P \to A_i$, such that for any object C and family $(\alpha_i)_{i\in I}$ of morphisms, $\alpha_i: C \to A_i$, there exists a *unique* morphism $\alpha: C \to P$ such that $\alpha_i = \varpi_i \alpha$. If the product P exists, it is unique up to isomorphism, as can be readily checked (or will be seen below as a particular case of the unicity of limits); it will be written $\prod_{i\in I} A_i$. Also note that the product of the empty family is the terminal object if it exists.

If $(A_i)_{i\in I}$ is a family of objects of a category $\mathbf{C}$, a coproduct of the family is an object Q, together with a family $(\iota_i)_{i\in I}$ of morphisms, $\iota_i: A_i \to Q$, such that for any object C and family $(\alpha_i)_{i\in I}$ of morphisms, $\alpha_i: A_i \to C$, there exists a *unique* morphism $\alpha: Q \to C$ such that $\alpha_i = \alpha\iota_i$. If the coproduct Q exists, it is unique up to isomorphism, as can be readily checked (or will be seen below as a particular case of the unicity of colimits); it will be written $\coprod_{i\in I} A_i$. Also note that the coproduct of the empty family is the initial object if it exists.

Arbitrary products and coproducts exist in the category **Set**: products are just ordinary cartesian products, and coproducts are disjoint unions.

In the category **PSet**, products are also cartesian products, and coproducts are (not so) disjoint unions with base points identified.

In the category **Group**, products are ordinary cartesian products, and coproducts are free products. In the category **AbGroup**, products are ordinary cartesian products, and coproducts are direct sums.

In the category **POSet**, products are cartesian products with the product order (that is, $(u_i) \leq (v_i)$ iff $u_i \leq v_i$ for all i), and coproducts are disjoint unions with the union order (elements are comparable only if they are in the same part).

If a partially ordered set is made into a category, then the product of a family of elements is just the greatest lower bound of the family if it exists, and the coproduct is similarily the least upper bound.

In the category **Ring**, products are ordinary cartesian products; coproducts exist, but I do not know their name ("braided tensor products" or some such thing). In the category **ComRing**, products are ordinary cartesian products, and coproducts are tensor products. In the category **Field**, products do not generally exist, and neither do coproducts.

If G is a group, then in G**Set**, products are ordinary cartesian products, and coproducts are disjoint unions, both with the obvious G-set structure.

The coproduct in **Scheme** is just the disjoint union. The product is the product of schemes as it is usually defined (the fiber product over $\operatorname{Spec}\mathbb{Z}$).

Products in a slice category are called "pullbacks", and we will describe them presently.

If J is an object of a category $\mathbf{C}$, $(A_i)_{i\in I}$ a family of objects of $\mathbf{C}$, and $(\varphi_i)_{i\in I}$ a family of morphisms, $\varphi_i: A_i \to J$, a pullback (or fiber product) of the family over J is an object P, together with a family $(\varpi_i)_{i\in I}$ of morphisms, $\varpi_i: P \to A_i$, such that $\varphi_i\varpi_i$ does not depend on i, and that for any object C and family $(\alpha_i)_{i\in I}$ of morphisms, $\alpha_i: C \to A_i$ such that $\varphi_i\alpha_i$ does not depend on i, there exists a *unique* morphism $\alpha: C \to P$ such that $\alpha_i = \varpi_i\alpha$. If the pullback P exists, it is unique up to isomorphism, as can be readily checked (or will be seen below as a particular case of the unicity of solutions to universal problems). By far the most important example is the case where I has just two elements, say $I = \{1, 2\}$. In that case, the pullback is generally written $P = A_1 \times_J A_2$; one also says that ϖ_2 is the pullback of φ_1 along φ_2 (and, of course, that ϖ_1 is the pullback of φ_2 along φ_1).

If $\varphi: A \to J$ is a morphism, its kernel pair is the pullback of A with itself over J, or, more precisely, the canonical maps $\varpi_1, \varpi_2: P \to A$ from the pullback $P = A \times_J A$. It is easy to check that both these maps are the identity on A (or more precisely, that the identity $1_A: A \to A$ taken twice provides a pullback of A with itself over J) iff φ is mono.

If J is an object of a category $\mathbf{C}$, $(A_i)_{i\in I}$ a family of objects of $\mathbf{C}$, and $(\varphi_i)_{i\in I}$ a family of morphisms, $\varphi_i: J \to A_i$, a pushout of the family over J is an object Q, together with a family $(\iota_i)_{i\in I}$ of morphisms, $\iota_i: A_i \to Q$, such that $\iota_i\varphi_i$ does not depend on i, and that for any object C and family $(\alpha_i)_{i\in I}$ of morphisms, $\alpha_i: A_i \to C$ such that $\alpha_i\varphi_i$ does not depend on i, there exists a *unique* morphism $\alpha: Q \to C$ such that $\alpha_i = \alpha\iota_i$. If the pushout Q exists, it is unique up to isomorphism, as can be readily checked (or will be seen below as a particular case of the unicity of solutions to couniversal problems). By far the most important example is the case where I has just two elements, say $I = \{1, 2\}$. In that case, the pushout is generally written $P = A_1 \amalg_J A_2$; one also says that ι_2 is the pushout of φ_1 along (or through) φ_2 (and, of course, that ι_1 is the pushout of φ_2 along φ_1).

If $\varphi: J \to A$ is a morphism, its cokernel pair is the pushout of A with itself over J, or, more precisely, the canonical maps $\iota_1, \iota_2: A \to Q$ to the pushout $Q = A \amalg_J A$. It is easy to check that both these maps are the identity on A (or more precisely, that the identity $1_A: A \to A$ taken twice provides a pushout of A with itself over J) iff φ is epi.

In **Set**, the pullback of a family $(A_i)_{i \in I}$ of sets over a set J, relative to the maps $\varphi_i: A_i \to J$, is the subset P of the cartesian product which consists of families $(x_i)_{i \in I}$ such that $\varphi_i(x_i)$ does not depend on i. In particular, the pullback $A \times_J B$ of two sets relative to $\varphi: A \to J$ and $\psi: B \to J$ is $A \times_J B = \{(x, y): \varphi(x) = \psi(y)\}$. And the pullback of φ through ψ is the projection of that set on the second coordinate. The kernel pair of $\varphi: A \to J$ is $\{(x, y): \varphi(x) = \varphi(y)\}$. And indeed, φ is mono iff the canonical maps $\varpi_1, \varpi_2: A \times_J A \to A$ are isomorphisms. The pushout of a family $(A_i)_{i \in I}$ of sets over a set J, relative to the maps $\varphi_i: J \to A_i$, is the quotient Q of the disjoint union of the A_i by the smallest equivalence relation which identifies all the $\varphi_i(t)$ for each $t \in J$. The cokernel pair of $\varphi: J \to A$ is the pair of canonical maps ι_1, ι_2 from A to the union of two copies of A with the $\varphi(J)$ identified. Obviously, ι_1 and ι_2 are isomorphisms iff φ is epi.

In the category **Top**, the pullback $A \times_J B$ of two topological spaces A and B over J relative to the continuous maps $\varphi: A \to J$ and $\psi: B \to J$, is the subspace $\{(x, y): \varphi(x) = \psi(y)\}$ of the product. The pushout $A \amalg_J B$ of two topological spaces A and B over J relative to the continuous maps $\varphi: J \to A$ and $\psi: J \to B$, is the quotient of the disjoint union by the smallest equivalence relation which identifies $\varphi(x)$ and $\psi(x)$ for all x. The pullback is the same in the category **HausTop**; only note that the subspace in question is closed in the cartesian product. In particular, the pullback of two (Hausdorff) compact spaces is compact. Let us now consider the pushout in the particular case of the cokernel pair: let $\varphi: J \to A$ be a continuous map; now we can't just take the quotient of the disjoint union by the equivalence relation which identifies the two copies of $\varphi(J)$, because that equivalence relation might not be closed, and then the quotient wouldn't be Hausdorff. So we let F be the closure of the image of φ, and we let Q be the quotient of the disjoint union of two copies of A by the equivalence relation which identifies the two copies of F. The resulting space is Hausdorff because F is closed. We let $\iota_1, \iota_2: A \to Q$ be the canonical maps onto the two copies of A. It is easy to check that this gives indeed the pushout, because two continuous maps α_1, α_2 from A to a *Hausdorff* space which coincide on the image of φ must coincide on F. The reader will have recognized that this is the reasoning we used to determine what an epi map is. Indeed, φ is epi iff $F = A$, i.e. iff the image of φ is dense in A.

In the category **Group**, the pullback $A \times_J B$ of two groups (relative to, etc.) is the subgroup $\{(x, y): \varphi(x) = \psi(y)\}$ of the cartesian product. The pushout $A \amalg_J B$ of two groups relative to the homomorphisms $\varphi: J \to A$ and $\psi: J \to B$ is the quotient of the free product $A * B$ by the normal subgroup N generated by the $\varphi(t)\psi(t)^{-1}$ for $t \in J$: that is (a generalization of) the amalgam. In **AbGroup**, the pullback remains the same. The pushout $A \amalg_J B$ is now the quotient of $A \oplus B$ by the image of $(\varphi, -\psi)$.

Pullbacks and pushouts also exist in **POSet**: the pullback is the subset of the cartesian product where the morphisms coincide, and the pushout is obtained by taking the disjoint union and then identifying the images of J (this is harder to describe than it is to understand).

If a partially ordered set is made into a category, then pullbacks and pushouts are in no way interesting: a pullback is just the same as a product (that is, a greatest lower bound), and a pushout as a coproduct (that is, a least upper bound).

Pullbacks in the categories **Ring** and **ComRing** are what they ought to be. We shall not describe pushouts.

The pullback of two schemes A, B over a scheme S is what is normally refered to as the fiber product $A \times_S B$ of the schemes in question.

If $(\alpha_i)_{i \in I}$ is a family of morphisms, $\alpha_i: A \to B$, then an equalizer of the family is an object P together with a morphism $\varepsilon: P \to A$ such that $\alpha_i\varepsilon$ does not depend on i, and that for any morphism $\gamma: C \to A$ such that $\alpha_i\gamma$ does not depend on i, there exists a *unique* $\chi: C \to P$ such that $\gamma = \varepsilon\chi$. The equalizer is unique up to isomorphism provided it exists; moreover, the morphism ε in question is mono. That is, the equalizer can be identified with a subobject of A.

If $(\alpha_i)_{i\in I}$ is a family of morphisms, $\alpha_i\colon A\to B$, then a coequalizer of the family is an object Q together with a morphism $\delta\colon B\to Q$ such that $\delta\alpha_i$ does not depend on i, and that for any morphism $\gamma\colon B\to C$ such that $\gamma\alpha_i$ does not depend on i, there exists a *unique* $\chi\colon Q\to C$ such that $\gamma=\chi\delta$. The coequalizer is unique up to isomorphism provided it exists; moreover, the morphism δ in question is epi. That is, the coequalizer can be identified with a quotient object of A.

If $\alpha_i\colon A\to B$ are morphisms in **Set**, that is, maps of sets, then their equalizer can be identified with the subset of A consisting of all x such that all the $\alpha_i(x)$ are equal. The coequalizer can be identified with the equivalence relation on B generated by the pairs $(\alpha_i(t),\alpha_j(t))$ with $t\in A$ and $i,j\in I$.

In **Top**, the equalizer and coequalizer are the same as in **Set**, with the induced and quotient topologies respectively. In **HausTop**, the equalizer is yet again the same.

In **Group**, the equalizer of a family (α_i) of morphism is the subgroup consisting of the points where all the α_i coincide. The coequalizer is the quotient of B by the normal subgroup of B generated by the $\alpha_i(t)\,\alpha_j(t)^{-1}$. Both these facts remain valid in **AbGroup**, only they can be more conveniently stated: the equalizer is the intersection of the kernels of the $\alpha_i-\alpha_j$, and the coequalizer is the quotient by the sum of the images of the $\alpha_i-\alpha_j$.

The resemblance between, say, pullbacks and equalizers, is not fortuitous. Indeed, if $(A_i)_{i\in I}$ is a family of objects and $\varphi_i\colon A_i\to J$ are morphisms, then the pullback can be constructed in the following way (provided everything exists): first take the product $P^\bullet$ of the A_i. Let $\varpi_i^\bullet\colon P^\bullet\to A_i$ be the canonical maps. Then let $\varepsilon\colon P\to P^\bullet$ be the equalizer of the family $(\varphi_i\varpi_i)_{i\in I}$, and put $\varpi_i=\varpi_i^\bullet\varepsilon_i$. The object P together with the morphisms ϖ_i provides the required pullback. The dual construction yields pushouts as a result of a coproduct followed by a coequalizer.

Before closing this section, let us discuss one last thing. If $\alpha\colon A\to B$ is a morphism, a section of α is a morphism $\sigma\colon B\to A$ such that $\alpha\sigma=1_B$, and a retraction of α is a morphism $\rho\colon B\to A$ such that $\rho\alpha=1_A$. A morphism which has a section is called sectionnable, and one which has a retraction is called retractable; obviously, every sectionnable morphism is epi, and every retractable morphism is mono. A quotient object of an object A whose defining epimorphisms are sectionnable is called a section of A, and a subobject of A whose defining monomorphisms are retractable is called a retract of A. (Hopefully this double use of the word "section" will not cause confusion.) A category in which every mono is retractable is called semisimple. In the category **Set**, it is well known that every epi has a section (in fact, this is the Axiom of Choice) and that every mono has a retraction.

In the category **HausTop**, not every mono has a retract; in fact, not every embedding has a retraction; for example, the circle is not a retract of the disk. And not every epi has a section; for example, consider the subset $\{(x,y)\colon(x\geq 0\wedge y=1)\vee(x\leq 0\wedge y=-1)\}$ of the plane and its projection on the first coordinate. For a more interesting example, consider the bundle of unit vectors on the sphere: this is well known to have no section.

In the category **Group**, a subgroup A of a group B is a retract of B iff there exists a normal subgroup H of B such that B is a semidirect product of H by A. [Proof: Suppose $A\leq B$ and $\varphi\colon B\to A$ is such that $\varphi(x)=x$ for all $x\in A$. Let $H=\ker\varphi$. Then $H\trianglerighteq B$, $H\cap A=\{1\}$ [obvious], and $HA=B$ [because if $x\in G$ then $x=x\varphi(x)^{-1}\varphi(x)$, and $x\varphi(x)^{-1}\in H$ and $\varphi(x)\in A$]. This means that G is a semidirect product of H by A. Conversely, assume the latter. Then the canonical map $\varphi\colon B\to A$ is a retraction of the canonical embedding $A\to B$.] A quotient object B/H of B is a section of B iff B is a semidirect product of H by some subgroup A of B. [Proof: If the canonical $\varpi\colon B\to B/H$ has a section $\varphi\colon B/H\to B$, then let $A=\operatorname{im}\varphi$, and check that B is a semidirect product of H by A. Conversely, let φ be the canonical embedding of $B/H\cong A$ in B.]

Let us now consider retracts in **POSet**. Let B be an object of **POSet**, which for simplicity we will assume totally ordered. A subobject A of B is a retract of B iff the following conditions are satisfied: (i) The order on A is induced by the order on B (i.e. it is no coarser), and (ii) For every $m\in B$, the set $\{x\in A\colon x\geq m\}$ has an infemum *in* A. [Proof: Suppose A is a retract of B. Let $\alpha\colon A\to B$ be mono defining A as a subobject of B, that is, α is injective and order-preserving;

there exists ρ order-preserving such that $\rho\alpha = 1_A$. Then α is an isomorphism onto its image, and so A can be identified with the image of α, which we do. This proves (i). Now let $m \in B$, and set $S = \{x \in A: x \geq m\}$. Clearly, $\rho(m)$ is a lower bound for S. Suppose now s is a lower bound for S. If $s \geq \rho(m)$, then it is false that $\rho(s) < \rho(m)$, and so it is false that $s < m$ (because ρ is injective): in other words, $s \geq m$, that is, $s \in S$. But then s is the least element of S, so S has a greatest lower bound in A (namely s), and we are finished. So we may suppose that $s \leq \rho(m)$ for each lower bound s of S. But in that case, $\rho(m)$ is the greatest lower bound of S, so again S has a greatest lower bound. This proves (ii). Now for the converse: suppose (i) and (ii). For every $m \in A$, define $\rho(m)$ to be the infemum of $S(m) := \{x \in A: x \geq m\}$, which we know to be in A. If $m \leq m'$ then $S(m) \supseteq S(m')$ so $\rho(m) \leq \rho(m')$, which means that ρ is order-preserving. And if $m \in A$ then obviously $\rho(m) = m$. So ρ is the desired retraction.]

If A and B are objects of R**Mod**, then $\alpha: A \to B$ mono is retractable iff the exact sequence $1 \to A \to B \to B/\alpha(A) \to 1$ splits, i.e. iff $\alpha(A)$ is a direct summand of B. In other words, the retracts of an R-module are exactly its direct summands. And so the category R**Mod** is semisimple iff the ring R is.

Let $A \subseteq B$ be two topological spaces, $\alpha: A \to B$ the canonical embedding of A in B. We have defined A to be a retract of B iff α is retractable in **Top**. A weaker statement is that (the homotopy class of) α be retractable in **HomoTop**. In that case, one says that A is a <u>weak retract</u> of B. Another interesting notion is that of a <u>deformation</u>: A is said to be a deformation of B iff α is sectionnable in **HomoTop**. A <u>weak deformation retract</u> is a subspace which is both a deformation and a weak retract, that is, a space whose canonical embedding is an isomorphism in the category **HomoTop**.

2. Functors.

Let $\mathbf{C}, \mathbf{D}$ be two categories. A <u>functor</u> $\mathfrak{F}$ from $\mathbf{C}$ to $\mathbf{D}$ consists of: a map $A \mapsto \mathfrak{F}(A)$ of the objects of $\mathbf{C}$ to the objects of $\mathbf{D}$, and for any two objects A, B of $\mathbf{C}$, a map $\alpha \mapsto \mathfrak{F}(\alpha)$ of $\mathrm{Hom}(A, B)$ to $\mathrm{Hom}(\mathfrak{F}(A), \mathfrak{F}(B))$, such that $\mathfrak{F}(1_A) = 1_{\mathfrak{F}(A)}$ for each object A of $\mathbf{C}$, and $\mathfrak{F}(\alpha\beta) = \mathfrak{F}(\alpha)\,\mathfrak{F}(\beta)$ whenever this makes sense. One generally writes $\mathfrak{F}: \mathbf{C} \rightsquigarrow \mathbf{D}$ to say that $\mathfrak{F}$ is a functor from $\mathbf{C}$ to $\mathbf{D}$.

The map which takes a pointed set (or group, or topological space, or partially ordered set) to the underlying set, and a morphism of pointed sets (or groups, or...) to the underlying map of sets, is a functor from the category **PSet** (or **Group**, or **Top**, or **POSet**) to the category **Set**. It is called the "forgetful" functor since it strips the given object of whatever structure it may have had. There is also a forgetful functor from **Group** to **PSet** which takes a group to its underlying pointed set, the base element being the unit element of the group; a forgetful functor from **Field** to **ComRing** for example, and so on.

Let A be any object in a category $\mathbf{C}$, and consider the map which takes an object B of $\mathbf{C}$ to the object $\mathrm{Hom}(A, B)$ of **Set**, and any morphism $\beta: B \to C$ to the map $\alpha \mapsto \beta\alpha$. This is a functor from $\mathbf{C}$ to **Set**, written $\mathrm{Hom}(A, \cdot)$.

Let A be a set. Consider the map which takes a set B to the set $A \times B$, and a map $\beta: B \to C$ to the map $(1_A, \beta): A \times B \to A \times C$. This is a functor from **Set** to **Set**, written $A \times \cdot$.

If two partially ordered sets are viewed as categories, then a functor between these can be identified with an order-preserving map of the posets.

If two groups are viewed as categories (with just one object), then a functor between these can be identified with a group homomorphism.

The <u>homology functor</u> is the functor H: **Hom∂AbGroup** $\rightsquigarrow$ **GradAbGroup** which takes a complex (K_n, ∂_n) to the sequence $(H_n K)$ of homology groups defined by $H_n K = \ker \partial_n / \operatorname{im} \partial_{n+1}$, and the homotopy class of a chain map (f_n) to the sequence of induced morphisms (a straightforward but lengthy calculation shows that the induced morphisms do not depend on the choice of the chain map in the homotopy class).

One domain of mathematics where functors play a very important role is algebraic topology. Most important of all are the <u>homotopy functor</u> π: **PHomTop** $\rightsquigarrow$ **GradGroup** which takes a pointed topological space to its sequence of homotopy groups, and the <u>singular complex functor</u> S: **HomTop** $\rightsquigarrow$ **Hom∂AbGroup** which takes a topological space to its singular complex (actually, it is quite useful to consider the singular complex of a pair of spaces, but we don't want to dwell on the details to much). Most important is the composite of the homology functor H: **Hom∂AbGroup** $\rightsquigarrow$ **GradAbGroup** with the singular complex functor S: **HomTop** $\rightsquigarrow$ **Hom∂AbGroup**, giving the singular homology functor HS: **HomTop** $\rightsquigarrow$ **GradAbGroup** (often written H instead of HS).

A functor $\mathfrak{F}$ is said to be full (resp. faithful) iff the map $\mathrm{Hom}(A,B) \to \mathrm{Hom}(\mathfrak{F}(A),\mathfrak{F}(B))$ it defines is surjective (resp. injective) for any pair of objects A, B.

Note that a full and faithful functor $\mathfrak{F}$ is essentially injective in the following sense: if $\mathfrak{F}(A)$ and $\mathfrak{F}(B)$ are isomorphic, then so are A and B. A functor $\mathfrak{F}$ from a category **C** to a category **D** is said to be essentially surjective iff for each object D of **D** there is an object A of **C** such that $\mathfrak{F}(A)$ and D are isomorphic.

A functor $\mathfrak{F}$: **C** $\rightsquigarrow$ **D** which is full, faithful and essentially surjective is said to be an equivalence of categories, and the categories **C** and **D** are then said to be equivalent.

A subcategory of a category **C** is a category **D** whose objects and morphisms are included in those of **C**. There is then a canonical injection functor from **C** to **D**; and **D** is said to be a full subcategory of **C** iff this injection functor is full (it is always faithful and essentially injective, of course).

The various forgetful functors we described are all faithful [in other words, a morphism in **PSet**, **Group**, **Top** or **POSet** is determined uniquely by its values on the points of its range]. None of them are full [any map need not be a morphism]. The forgetful functors from **Group**, **Top** and **POSet** to **Set** are not essentially injective [two equipotent structures need not be isomorphic]. The forgetful functor from **PSet** to **Set**, however, is essentially injective [two pointed sets are isomorphic iff they have same cardinality]. The forgetful functors from **Top** and **POSet** to **Set** are essentially surjective [any set can be made into a topological set or can be partially ordered], in fact *actually* surjective if need be. The forgetful functor from **Group** to **PSet** is essentially surjective [any pointed set can be made into a group with the base element as an identity element]; but the forgetful functor from **Group** to **Set** is not essentially surjective, and for a stupid reason: a group cannot be empty.

The functor $\mathrm{Hom}(A,\cdot)$ from a category **C** to **Set** does not generally have any of the above mentioned properties.

If A is a set, the functor $A\times\cdot$ is faithful iff A is non empty, is full iff A has exactly one element (note that when A is empty, the functor is not full, since C^B is empty when C is and B is not, whereas $(A\times C)^{(A\times B)}$ is not empty; this comment is completely devoid of any interest, of course). Moreover, the functor in question is essentially injective iff A is finite, essentially surjective iff A has exactly one element.

Consider, for $0\leq r\leq\omega$ the category C^r**Man** of C^r-manifolds. If $s\leq r$, then there is a forgetful functor from C^r**Man** to C^s**Man**. This functor is faithful, and is not full unless $s=r$. When $s=0$ and $r>0$, it is known (though by no means obvious) that the functor is neither essentially injective [there exist two C^r-manifolds which are homeomorphic without being C^r-diffeomorphic] nor essentially surjective [there exists a C^0-manifold which does not admit any C^r-manifold structure]. When $s>0$, on the other hand, the functor *is* essentially injective *and* essentially surjective, as a well-known approximation theorem states.

The functor $\mathfrak{F}$: **Top** $\rightsquigarrow$ **HomoTop** which is the identity on the objects, and takes a continuous map to its homotopy class, is full but not faithful. It is essentially surjective (indeed, surjective), but not essentially injective.

Let us also note that **AbGroup** is a full subcategory of **Group**, and **HausTop** a full subcategory of **Top**.

If **C** and **D** are categories, then a contravariant functor from **C** to **D** is just a functor from $\mathbf{C}^{\mathbf{op}}$ to **D**. When the need arises, a functor from **C** to **D** can be called a "covariant functor" to distinguish it from a contravariant functor.

If $\mathbf{C}_1,\ldots,\mathbf{C}_n$ are categories, then one can define a product category $\mathbf{C} := \mathbf{C}_1\times\cdots\times\mathbf{C}_n$ in the following (obvious) way: objects in **C** are n-tuples $(A_1,\ldots,A_n)$ with A_i an object of $\mathbf{C}_i$ for each i, and $\mathrm{Hom}((A_1,\ldots,A_n),(B_1,\ldots,B_n))$ is just $\mathrm{Hom}(A_1,B_1)\times\cdots\times\mathrm{Hom}(A_n,B_n)$, with composition defined componentwise.

A functor from a product category is called a functor of several variables. Most important is the case when all the factors are the same category or its opposite: the functor is then said to be covariant of so-many variables and contravariant of so-many.

There is a covariant functor of two variables on **Set**, which is defined in the following way: it takes a pair of sets (A, B) to $A \times B$, and a pair of maps (α, β) to the map $(x, y) \mapsto (\alpha(x), \beta(y))$.

There is a functor on **Set** contravariant in one variable and covariant in another one, defined in the following way: it takes a pair of sets (A, B) to B^A, and a pair of maps (α, β) to $f \mapsto \beta f \alpha$.

More generally, on any category **C**, the functor $\mathrm{Hom}(\cdot, \cdot)$ (defined in a similar way), is contravariant in the first variable, and covariant in the second.

Let R be a commutative ring; then the following functor $\mathfrak{F}$ is contravariant on R**Mod**: $\mathfrak{F}(A)$ is the dual of A for any R-module A, and $\mathfrak{F}(\alpha)$ is the transpose of α for any R-linear map of R-modules. The composite* of this functor with itself gives a covariant functor $\mathfrak{G}$ on R**Mod**: $\mathfrak{G}(A)$ is the bidual of A for any R-module A, and $\mathfrak{G}(\alpha)$ is the bitranspose of α for any R-linear map of R-modules.

The map which takes any commutative ring A to its spectrum $\mathrm{Spec}\, A$, and a ring homomorphism $\alpha: A \to B$ to the associated regular map ${}^a\alpha: \mathrm{Spec}\, B \to \mathrm{Spec}\, A$, is a contravariant functor from **ComRing** to **Scheme**.

Before we close this section, let us mention one last definition, which can be useful under some circumstances: a partial functor from **C** to **D** is a functor from a subcategory of **C** to **D**.

3. Natural transformations.

Let **C** and **D** be two categories, and $\mathfrak{F}, \mathfrak{G}: \mathbf{C} \rightsquigarrow \mathbf{D}$. Then a natural transformation (or natural map) from $\mathfrak{F}$ to $\mathfrak{G}$ is a map α which to every object A of **C** associates a morphism $\alpha(A): \mathfrak{F}(A) \to \mathfrak{G}(A)$, in such a way that for every morphism $\varphi: A \to B$ of **C** we have

$$\mathfrak{G}(\varphi)\, \alpha(A) = \alpha(B)\, \mathfrak{F}(\varphi)$$

We will write $\alpha: \mathfrak{F} \to \mathfrak{G}$.

A natural map from a contravariant functor $\mathfrak{F}$ to a contravariant functor $\mathfrak{G}$ is defined in the obvious way, since a contravariant functor from **C** to **D** is the same as a (covariant) functor from $\mathbf{C}^{\mathbf{op}}$ to **D**. The same applies to functors of several variables.

If $\mathfrak{F}, \mathfrak{G}, \mathfrak{H}$ are three functors from a category **C** to a category **D**, and $\alpha: \mathfrak{F} \to \mathfrak{G}$ and $\beta: \mathfrak{G} \to \mathfrak{H}$ are natural maps, then there is a natural map $\beta\alpha: \mathfrak{F} \to \mathfrak{H}$ defined in the following way: $(\beta\alpha)(A) = \beta(A)\, \alpha(A)$ for each object A of **C**.

Moreover, if $\mathfrak{F}: \mathbf{C} \rightsquigarrow \mathbf{D}$, then there is a natural map $1_{\mathfrak{F}}: \mathfrak{F} \to \mathfrak{F}$ defined by $1_{\mathfrak{F}}(A) = 1_{\mathfrak{F}(A)}$ for every object A of **C**. If **C** and **D** are two categories, then the class of all functors from **C** to **D**, with natural maps as morphisms, forms a category. That category is usually written $\mathbf{D}^{\mathbf{C}}$.

A natural equivalence is a natural map α such that $\alpha(A)$ is an isomorphism for every object A It is easily checked that $\alpha: \mathfrak{F} \to \mathfrak{G}$ is a natural equivalence iff α is an isomorphism when viewed as a morphism of $\mathbf{D}^{\mathbf{C}}$. When there exists a natural equivalence between two functors, they are said to be naturally equivalent.

Let R be a ring, and consider the functor $\mathfrak{F}$ from R**Mod** to R**Mod** which takes a module to its bidual, and a morphism to its bitranspose. Let $\mathfrak{I}$ be the identity functor on R**Mod**. And lastly, for each R-module A, let $\alpha(A)$ be the map which takes $x \in A$ to the element $l \mapsto l(x)$ of the bidual of A. Then $\alpha: \mathfrak{I} \to \mathfrak{F}$ is a natural map.

Consider the functor $\mathfrak{F}: \mathbf{Set} \times \mathbf{Set}^{\mathbf{op}} \times \mathbf{Set}^{\mathbf{op}} \rightsquigarrow \mathbf{Set}$ which takes (A, B, C) to $A^{B \times C}$, and the functor $\mathfrak{G}: \mathbf{Set} \times \mathbf{Set}^{\mathbf{op}} \times \mathbf{Set}^{\mathbf{op}} \rightsquigarrow \mathbf{Set}$ which takes (A, B, C) to $(A^B)^C$ (the action of F and G on morphisms is defined in the obvious way, see below). For any three objects (A, B, C) of **Set**, let $\alpha(A, B, C)$ be the following map from $A^{B \times C}$ to $(A^B)^C$:

$$f \mapsto (z \mapsto f(\cdot, z))$$

* Of course, this is not quite correct. $\mathfrak{F}$ goes from $R\mathbf{Mod}^{\mathbf{op}}$ to R**Mod**. So it cannot be composed with itself, but rather with the evident functor $\mathfrak{F}^{\mathbf{op}}$.

Then $\alpha: \mathfrak{F} \to \mathfrak{G}$ is a natural equivalence. [Proof: Let $\varphi: A \to A'$, $\chi: B' \to B$ and $\psi: C' \to C$ be maps of sets. Then $\mathfrak{F}(\varphi, \chi, \psi)$ is the map $f \mapsto ((y,z) \mapsto \varphi(f(\chi(y), \psi(z))))$ and $\mathfrak{G}(\varphi, \chi, \psi)$ is the map $f \mapsto (z \mapsto (y \mapsto \varphi(f(\psi(z))(\chi(y)))))$. So $\mathfrak{G}(\varphi, \chi, \psi)\, \alpha(A, B, C)$ is:

$$f \mapsto (z \mapsto (y \mapsto \varphi(f(\chi(y), \psi(z)))))$$

whereas $\alpha(A', B', C')\, \mathfrak{F}(\varphi, \chi, \psi)$ is:

$$f \mapsto (z \mapsto (y \mapsto \varphi(f(\chi(y), \psi(z)))))$$

i.e. the same. So α is a natural map. Moreover, $\alpha(A, B, C)$ is bijective for every A, B, C. This concludes the (obvious) proof.]

Statement: there does not exist a natural map $\alpha: \mathfrak{I} \to \mathfrak{I}$, where $\mathfrak{I}$ is the identity functor on **Set**, such that $\alpha(A): A \to A$ is constant for every set A. [Proof: Suppose otherwise, and let A be a set with two elements, $\varphi: A \to A$ the map exchanging those two elements. Since α is assumed natural, we ought to have $\varphi\alpha(A) = \alpha(A)\,\varphi = \alpha(A)$ (the last equality because $\alpha(A)$ is constant), which is a contradiction. This concludes the proof.] In other words, there exists no "canonical" way of choosing one element from each non empty set. On the other hand: there does exist exactly one natural map $\alpha: \mathfrak{I} \to \mathfrak{I}$ where $\mathfrak{I}$ is the identity functor on **PSet** such that $\alpha(A): A \to A$ is constant for every pointed set A. [Proof: Obviously, $\alpha(A)$ must be the constant map equal to the base element of A. So let it be so. We must prove that for every pointed set A and morphism $\varphi: A \to A$, we have $\varphi\alpha(A) = \alpha(A)\,\varphi$. But the left hand side is the constant map equal to the base element of A, and so is the right hand side, since φ takes the base element to the base element.] In other words, there exists a canonical way of choosing an element from each pointed set. Other statement: if $\mathfrak{F}$ is the forgetful functor from **PSet** to **Set**, then there exists a natural map $\alpha: \mathfrak{F} \to \mathfrak{F}$ such that for every pointed set A, the map $\alpha(A): \mathfrak{F}(A) \to \mathfrak{F}(A)$ is constant. [Proof: Let $\alpha(A)$ be the constant map equal to the element of $\mathfrak{F}(A)$ which is the base element of A. The rest of the proof is similar to the previous.] Exercice: show that this is the only such map.

Let **C** be a category consisting of sets with some extra structure on them (e.g. groups, abelian groups, topological spaces, partially ordered sets, etc.), and let A be an object of **C**. Define **A** to be the full subcategory of **C** consisting of the sole object A (so the morphisms are exactly the endomorphisms of A in **C**). Let $\mathfrak{G}$ be the forgetful functor from **A** to **Set**. Then a natural map $\alpha: \mathfrak{G} \to \mathfrak{G}$ can be identified with a map $\hat{\alpha} := \alpha(A): \mathfrak{G}(A) \to \mathfrak{G}(A)$, that is, a map of sets of A into itself, which is such that $\varphi\hat{\alpha} = \hat{\alpha}\varphi$, as maps of sets, for every endomorphism φ of A. That is, $\hat{\alpha}$ is a map of the base set of A into itself which is invariant under all endomorphisms of A. Note that $\hat{\alpha}$ is not required to be a *morphism*, merely a map of sets. (To require $\hat{\alpha}$ to be a morphism, we would just take the identity functor $\mathfrak{G}$ on **A** instead of the forgetful functor.) If instead of taking **A** to be a full subcategory of **C** we just kept automorphisms of A as morphisms, we would get the (weaker) condition that $\hat{\alpha}$ is invariant under all automorphisms of A. If $\hat{\alpha}$ is constant, always equal to an element $\bar{\alpha}$ of A (or, more rigorously, of $\mathfrak{G}(A)$), then saying that α is natural means that $\bar{\alpha}$ is invariant under all automorphisms of A.

If $\mathfrak{F}$ and $\mathfrak{G}$ are two partial functors from a category **C** to a category **D**, then a natural map $\mathfrak{F} \to \mathfrak{G}$ is just a natural map defined on the objects of the intersection of the subcategories of **C** over which $\mathfrak{F}$ and $\mathfrak{G}$ are defined.

4. Universal and couniversal problems. Limits and colimits.

If $\mathbf{C}$ is a category, the solution to the <u>universal problem</u> posed by the category $\mathbf{C}$ is simply the terminal object of $\mathbf{C}$ if it exists. The solution to the <u>couniversal problem</u> posed by the category $\mathbf{C}$ is simply the initial object of $\mathbf{C}$ if it exists.

Let $\mathfrak{G}\colon \mathbf{C} \rightsquigarrow \mathbf{D}$ be a functor, and B an object of $\mathbf{D}$. Let $\mathbf{B}$ be the category whose objects are pairs (C,β) with C an object of $\mathbf{C}$ and $\beta\colon B \to \mathfrak{G}(C)$ a morphism in $\mathbf{D}$, and whose morphisms $\alpha \in \mathrm{Hom}_{\mathbf{B}}((C,\beta),(C',\beta'))$ are the elements of $\mathrm{Hom}_{\mathbf{C}}(C,C')$ such that $\mathfrak{G}(\alpha)\beta = \beta'$. In other words, $\mathbf{B}$ is the category of objects of $\mathbf{C}$ "around" B. The solution to the couniversal problem posed by $\mathbf{B}$ (if it exists) is called the free object of $\mathbf{C}$ with basis B with respect to the functor $\mathfrak{G}$. For example, $\mathfrak{G}$ might be the forgetful functor from **Group** to **Set**, in which case we get the classical notion of a free group with basis B. Similarily, we get the notion of free abelian group with basis B, or of free R-module with basis B (where R is any ring). If $\mathfrak{G}\colon \mathbf{ComRing} \rightsquigarrow \mathbf{Set}$ is the forgetful functor, then the free object with basis B is the ring of polynomials with integer coefficients and the elements of B as unknowns. With **Ring** instead of **ComRing**, we get the ring of polynomials with integer coefficients and *noncommuting* variables. If now instead of taking a "completely forgetful" functor, we take a "partially forgetful" functor. Say the forgetful functor $\mathfrak{G}\colon \mathbf{ComRing} \rightsquigarrow \mathbf{AbGroup}$ which forgets multiplication: in this case the couniversal problem does indeed have a solution, but I do not believe it has a classical name. It may be easily described for finitely generated abelian groups, however: indeed, first notice that if $B = B' \oplus B''$ then the free ring on B (with respect to the functor $\mathfrak{G}$) is the tensor product over $\mathbb{Z}$ of the free rings on B' and B''; so it remains only to describe the free ring on $\mathbb{Z}$ and on $\mathbb{Z}/n\mathbb{Z}$: the former is just $\mathbb{Z}[X]$, and the second is $\mathbb{Z}[X]/(nX)$. One could also let **SemiGroup** be the category of semigroups, and $\mathfrak{G}\colon \mathbf{Ring} \rightsquigarrow \mathbf{SemiGroup}$ be the functor which forgets addition. In this case, the free ring on a semigroup B is what is usually called the semigroup ring $\mathbb{Z}B$. Finally, we might also take a group G and consider the forgetful functor $\mathfrak{G}\colon G\mathbf{Set} \rightsquigarrow \mathbf{Set}$. Then the "free G-set" with basis a set B is the G-set $G \times B$ with G acting by multiplication on the left on the first coordinate (G then acts freely, hence the name).

The example given in the previous paragraph also has many important applications in topology. Let **ComTop** be the category of (Hausdorff) topological spaces, and $\mathfrak{G}\colon \mathbf{ComTop} \rightsquigarrow \mathbf{Top}$ be the inclusion functor. Then, for B an object of **Top**, the "free compact space with basis B" (this terminology is very unappropriate in this case) is the Stone-Čech compactification of B, sometimes written βB. As another application, we let $\mathfrak{G}\colon T_n\mathbf{Top} \rightsquigarrow \mathbf{Top}$ be the inclusion functors, where $T_0\mathbf{Top}$ is the category of T_0 (Kolmogoroff) spaces, $T_1\mathbf{Top}$ the category of T_1 (accessible) spaces, $T_2\mathbf{Top} = \mathbf{HausTop}$ the category of T_2 (Hausdorff) spaces, and $T_3\mathbf{Top}$ the category of T_3 (regular) spaces. Then, for B a topological space, the universal problem mentioned above has a solution, i.e. there exists a "free T_n space with basis B" (this terminology is extremely unappopriate), called the universal T_n space of B. (For a proof, see Bourbaki, General Topology, chapter I, exercice 27 to section 8.)

As a somewhat more far-fetched example of the same construction, we let $\mathbb{C}\mathbf{LieAlg}$ be the category of complex Lie algebras, $\mathbb{C}\mathbf{Alg}$ the category of complex associative algebras, and $\mathfrak{G}\colon \mathbb{C}\mathbf{Alg} \rightsquigarrow \mathbb{C}\mathbf{LieAlg}$ the (more or less) forgetful functor which makes an associative algebra A into a Lie algebra by keeping just addition, scalar multiplication, and the bracket $[u,v] = uv - vu$ (the action of $\mathfrak{G}$ on morphisms is just what you think). Then the "free associative algebra with basis a Lie algebra B" is what is normally called the universal envelopping algebra $U(B)$ of B.

If R is a ring, and A, B two R-modules, let $\mathbf{C}$ be the category whose objects are *R-bilinear* maps $\varphi\colon A \times B \to C$ with C an abelian group, and whose morphisms $\alpha\colon \varphi \to \psi$ are morphisms $\alpha\colon C \to D$ in **AbGroup** such that $\psi = \alpha\varphi$. Then the solution to the couniversal problem posed by $\mathbf{C}$ exists, and is the tensor product $A \otimes_R B$.

We saw above how the Stone-Čech compactification can be considered as the solution of a couniversal problem. It is interesting to note that the other major compactification procedure, namely the Alexandroff (one-point) compactification, is solution of a *universal* problem. Indeed, let A be a (Hausdorff) locally compact topological space, that is *not* compact. Let $\mathbf{A}$ be the category whose objects are *open dense embeddings* of A into a compact space C (i.e. homeomorphisms of A onto an open dense subspace of a compact space C), with morphisms defined in the now familiar way: a morphism from $\varphi: A \to C$ to $\psi: A \to D$ is a continuous map $\alpha: C \to D$ such that $\psi = \alpha\varphi$. Then the solution to the universal problem posed by $\mathbf{A}$ is precisely the Alexandroff compactification of A. [Proof: Let D be the Alexandroff compactification of A; we consider A as a subspace of D, and we let ω be the point at infinity. Since A is not compact, it cannot be closed in D, and hence it is dense; moreover, since D is Hausdorff hence T_1, the point $\{\omega\}$ is closed in D, so A is open in D; therefore, the canonical embedding $\psi: A \to D$ is an object of $\mathbf{A}$. Now let $\varphi: A \to C$ be any open dense embedding of A into a compact space C. We view again A as a subspace of C. We have to prove that there exists a unique $\alpha: C \to D$ such that $\alpha|_A$ is the identity on A. Let α be the map which is the identity on A and which takes any other point of C to ω. Let us prove that α is continuous: let Ω be open in D; either $\Omega \subseteq A$, in which case $\alpha^{-1}(\Omega)$ is obviously open since A is; else $\omega \in \Omega$, in which case $D \setminus \Omega$ is compact, hence closed in C, and therefore $\alpha^{-1}(\Omega)$ is again open. Therefore α is continuous. Now if $\alpha': C \to D$ is another continuous map which is the identity on A, then α and α' coincide on the dense subset A of C, so they coincide on C. This proves what we wanted.]

Now let A be a commutative ring, and B a multiplicative subset of A (i.e. B contains 1 and is closed under multiplication). Let $\mathbf{B}$ be the category whose objects are morphisms $\varphi: A \to C$ in **ComRing** with $\varphi(x)$ invertible for each $x \in B$, and whose morphisms are defined as usual. Then the solution to the couniversal problem posed by $\mathbf{B}$ exists, and is the localization of A at B, written A_B or $B^{-1}A$. In case A is an integral domain, and B is the subset of its non zero elements, we get the familiar quotient field of A, which can also be described as the solution to the couniversal problem posed by the category whose objects are *injective* morphisms of A into a field.

Let A be a connected and locally simply connected topological space. Let $\mathbf{A}$ be the category whose objects are coverings of A, that is, continuous maps $\varphi: B \to A$ such that for every $x \in A$ there exists a neighbourhood V of x such that $\varphi^{-1}(V)$ is a nonempty disjoint union of subsets of B which are maped homeomorphically onto V, and whose morphisms $\alpha: \varphi \to \psi$ where $\varphi: B \to A$ and $\psi: C \to A$ are morphisms $\alpha: B \to C$ such that $\varphi = \psi\alpha$. Then the couniversal problem posed by the category $\mathbf{A}$ has a solution, namely the universal covering space of A.

Let R be a commutative ring, and A a commutative R-algebra. Let $\mathbf{A}$ be the category whose objects are derivations $d: A \to B$ with B an A-module, that is, R-linear maps such that $d(ab) = ad(b) + d(a)\,b$, and whose morphisms $\alpha: d \to d'$ are A-linear maps $B \to B'$ such that $d' = \alpha d$. Then the couniversal problem posed by $\mathbf{A}$ has a solution, called the universal derivation of $\mathbf{A}$.

Let $\mathbf{C}$ be a category, and $(A_i)_{i\in I}$ a family of objects of $\mathbf{C}$. Let $\mathbf{A}$ be the category whose objects are families $(\varphi_i)_{i\in I}$ of morphisms of $\mathbf{C}$, with $\varphi_i: B \to A_i$, and whose morphisms $\alpha: (\varphi_i) \to (\varphi'_i)$ are morphisms $\alpha: B \to B'$ of $\mathbb{C}$ such that $\varphi_i = \varphi'_i\alpha$. Then the solution to the universal problem posed by $\mathbf{A}$ is the same thing as the product of the (A_i) (that is, exists iff the product exists, and is then isomorphic to it). This is part of a more general construction, which we will describe presently.

Let $\mathbf{I}$ be a category (if we were to worry about set-theoretical difficulties, we would take $\mathbf{I}$ to be small, that is, such that the class of objects of $\mathbf{I}$ is a set), and $\mathfrak{F}: \mathbf{I} \rightsquigarrow \mathbf{C}$ a functor from $\mathbf{I}$ to an arbitrary category $\mathbf{C}$. A cone on $\mathfrak{F}$ with vertex C, where C is an object of $\mathbf{C}$, is a natural map $\mathfrak{C} \to \mathfrak{F}$, where $\mathfrak{C}: \mathbf{I} \rightsquigarrow \mathbf{C}$ takes all objects to C and all morphisms to 1_C; in other words, a cone with vertex C is a family $(\varphi_i)_{i\in\mathrm{ob}(\mathbf{I})}$ of objects of C, with $\varphi_i: C \to \mathfrak{F}(i)$, such that $\varphi_j = \mathfrak{F}(u)\,\varphi_i$ for each $u: i \to j$ in $\mathbf{I}$. If φ and ψ are cones on $\mathfrak{F}$ with vertices C and D respectively, then a morphism $\alpha: \varphi \to \psi$ is a morphism $\alpha: C \to D$ in $\mathbf{C}$ such that $\varphi_i = \psi_i\alpha$ for every object i of $\mathbf{I}$. Let $\mathbf{F}$ be the category of cones on $\mathfrak{F}$. In other words, $\mathbf{F}$ is the subcategory of the slice category $\mathbf{C}^{\mathbf{I}} \downarrow \mathfrak{F}$ consisting of natural maps from a constant functor. Then the limit (or projective limit) of the functor $\mathfrak{F}$ is the solution to the universal problem posed by $\mathbf{F}$ if it exists.

The notion of colimit is obtained by dualizing that of a limit: let $\mathbf{I}$ be a category, and $\mathfrak{F}: \mathbf{I} \rightsquigarrow \mathbf{C}$ a functor from $\mathbf{I}$ to an arbitrary category $\mathbf{C}$. A cocone on $\mathfrak{F}$ with vertex C, where C is an object of $\mathbf{C}$, is a natural map $\mathfrak{F} \to \mathfrak{C}$, where $\mathfrak{C}: \mathbf{I} \rightsquigarrow \mathbf{C}$ takes all objects to C and all morphisms to 1_C; in other words, a cocone with vertex C is a family $(\varphi_i)_{i\in\mathrm{ob}(\mathbf{I})}$ of objects of C, with $\varphi_i: \mathfrak{F}(i) \to C$, such that $\varphi_i = \varphi_j\mathfrak{F}(u)$ for each $u: i \to j$ in $\mathbf{I}$. If φ and ψ are cocones on $\mathfrak{F}$ with vertices C and D respectively, then a morphism $\alpha: \varphi \to \psi$ is a morphism $\alpha: C \to D$ in $\mathbf{C}$ such that $\psi_i = \alpha\varphi_i$ for every object i of $\mathbf{I}$. Let $\mathbf{F}$ be the category of cones on $\mathfrak{F}$. In other words, $\mathbf{F}$ is the subcategory of the coslice category $\mathfrak{F} \uparrow \mathbf{C}^{\mathbf{I}}$ consisting of natural maps to a constant functor. Then the colimit (or inductive limit) of the functor $\mathfrak{F}$ is the solution to the couniversal problem posed by $\mathbf{F}$ if it exists.

Let $\mathbf{I}$ be the category whose objects are the points of a certain set I, and with no morphism except the identities. The the limit of a functor $\mathfrak{F}: \mathbf{I} \rightsquigarrow \mathbf{C}$ is the product of the $\mathfrak{F}(i)$ for $i \in \mathbf{I}$, and the colimit of the same functor is the coproduct of the $\mathfrak{F}(i)$.

Let $\mathbf{I}$ be the category with just two objects, A and Ω, and whose morphisms are the elements of a set I, going from A to Ω, plus the two morphisms $1_A, 1_\Omega$ (composition is defined in the only possible way). Then a functor $\mathfrak{F}: \mathbf{I} \rightsquigarrow \mathbf{C}$ can be identified with a family $(\alpha_i)_{i \in I}$ of morphisms $\mathfrak{F}(A) \to \mathfrak{F}(\Omega)$, and the limit of $\mathfrak{F}$ is exactly the equalizer of the family (α_i). Similarily, the colimit of $\mathfrak{F}$ is the coequalizer of the family (α_i).

Let $\mathbf{I}$ be the category whose objects are the elements of a set I plus one extra object Ω, and which has, appart from the identity morphisms, exactly one morphism $i \to \Omega$ for each $i \in I$. Then a functor $\mathfrak{F}: \mathbf{I} \rightsquigarrow \mathbf{C}$ can be identified with an object J (namely $\mathfrak{F}(\Omega)$), a family $(A_i)_{i \in I}$ of objects (namely $\mathfrak{F}(i)$), and a family $\varphi_i: A_i \to J$ of objects. The limit of the functor $\mathfrak{F}$ is then exactly the pullback of the morphisms φ_i over the object J. Dually, the colimit of a functor $\mathfrak{F}$ on a category with one initial object and a family of other objects, defines the pushout of a family of morphisms.

Let now I be a partially ordered set (or even a preordered set) made into a category $\mathbf{I}$, as was previously described. Then a functor $\mathfrak{F}: \mathbf{I} \rightsquigarrow \mathbf{C}$ can be identified with a family $(A_i)_{i \in I}$ of objects of $\mathbf{C}$ and a family of morphisms $(f_{ji})_{i \leq j}$, $f_{ji}: A_i \to A_j$ such that $f_{ii} = 1_{A_i}$ for each i and that $f_{kj} f_{ji} = f_{ki}$ whenever $i \leq j \leq k$. The colimit of such a functor is what is classically called the inductive (or direct) limit of the A_i relative to the morphisms f_{ji} (at least when the set I satisfies the condition that for any $i, j \in I$ there exists $k \in I$ with $i \leq k$ and $j \leq k$). To define the projective (or inverse) limit, one will generally let $\mathbf{I}$ be the opposite category to the one given above, so that a functor $\mathfrak{F}: \mathbf{I} \rightsquigarrow \mathbf{C}$ is identified with a family $(A_i)_{i \in I}$ of objects of $\mathbf{C}$ and a family of morphisms $(f_{ij})_{i \leq j}$, $f_{ij}: A_j \to A_i$ such that $f_{ii} = 1_{A_i}$ for each i and that $f_{ij} f_{jk} = f_{ik}$ whenever $i \leq j \leq k$. The limit of such a functor is what is classically called the projective (or inverse) limit of the A_i relative to the morphisms f_{ij}.

Let us now discuss a very important universal property, which does not fall into the formalism of limits. Let $\mathbf{C}, \mathbf{D}$ be two categories, $\mathfrak{F}: \mathbf{C} \rightsquigarrow \mathbf{D}$ a functor. Consider the category $\mathbf{F}$ whose objects are natural maps $\varphi: \mathfrak{F}\mathfrak{G} \to \mathfrak{I}$ with $\mathfrak{I}$ the identity functor on $\mathbf{D}$ and $\mathfrak{G}: \mathbf{D} \rightsquigarrow \mathbf{C}$ a *partial* functor (recall that this means that $\mathfrak{G}$ is not necessarily defined over all of $\mathbf{D}$), and whose morphisms are natural transformations $\alpha: \mathfrak{G} \to \mathfrak{G}'$ such that[*] $\varphi = \varphi' \mathfrak{F}\alpha$. Then the solution to the universal problem posed by $\mathbf{F}$, if it exists and is defined on all of $\mathbf{D}$, is called the right adjoint to the functor $\mathfrak{F}$ (or, more precisely, the solution is a natural map $\mathfrak{F}\mathfrak{G} \to \mathfrak{I}$, called the counit of the adjunction, and $\mathfrak{G}$ is the right adjoint to $\mathfrak{F}$).

Note that if $\mathfrak{P}$ is right adjoint to $\mathfrak{F}$, $\varepsilon: \mathfrak{F}\mathfrak{P} \to \mathfrak{I}$ being the counit of the adjunction, then for every object D of $\mathbf{D}$, the morphism $\varepsilon(D)$ satisfies the following universal property: if C is an object of $\mathbf{C}$, and $\varphi: \mathfrak{F}C \to D$, then there exists a unique $\alpha: C \to \mathfrak{P}D$ such that $\varphi = \varepsilon(D)\,\mathfrak{F}\alpha$. To see that, simply let $\mathfrak{G}$ be the functor defined only on D and 1_D and taking them to C and 1_C respectively. Then a natural map $\mathfrak{G} \to \mathfrak{P}$ can be identified with a morphism $C \to \mathfrak{P}D$, and the universal property of the right adjoint gives the desired result.

Dually, there is the notion of a left adjoint, of course. Let $\mathbf{C}, \mathbf{D}$ be two categories, $\mathfrak{G}: \mathbf{D} \rightsquigarrow \mathbf{C}$ a functor. Consider the category $\mathbf{G}$ whose objects are natural maps $\varphi: \mathfrak{I} \to \mathfrak{G}\mathfrak{F}$ with $\mathfrak{I}$ the identity functor on $\mathbf{C}$ and $\mathfrak{F}: \mathbf{C} \rightsquigarrow \mathbf{D}$ a partial functor, and whose morphisms are natural transformations $\alpha: \mathfrak{F} \to \mathfrak{F}'$ such that $\varphi' = \mathfrak{G}\alpha\varphi$. Then the solution to the couniversal problem posed by $\mathbf{G}$, if it exists and is defined on all of $\mathbf{C}$, is called the left adjoint to the functor $\mathfrak{G}$ (or, more precisely, the solution is a natural map $\mathfrak{I} \to \mathfrak{G}\mathfrak{F}$, called the unit of the adjunction, and $\mathfrak{F}$ is the left adjoint to $\mathfrak{G}$).

If $\mathfrak{Q}$ is left adjoint to $\mathfrak{G}$, $\eta: \mathfrak{I} \to \mathfrak{G}\mathfrak{Q}$ being the unit of the adjunction, then for every object C of $\mathbf{C}$, the morphism $\eta(C)$ satisfies the following couniversal property: if D is an object of $\mathbf{D}$, and $\varphi: C \to \mathfrak{G}D$, then there exists a unique $\alpha: \mathfrak{Q}C \to D$ such that $\varphi = \mathfrak{G}\alpha\eta(C)$. To see that, simply let $\mathfrak{F}$ be the functor defined only on C and 1_C and taking them to D and 1_D respectively. Then a natural map $\mathfrak{Q} \to \mathfrak{F}$ can be identified with a morphism $\mathfrak{Q}C \to D$, and the universal property of the left adjoint gives the desired result.

[*] What is meant here by $\mathfrak{F}\alpha$ is the natural transformation from $\mathfrak{F}\mathfrak{G}$ to $\mathfrak{F}\mathfrak{G}'$ which takes A to the morphism $\mathfrak{F}(\alpha(A)): \mathfrak{F}\mathfrak{G}(A) \to \mathfrak{F}\mathfrak{G}'(A)$. The expression $\varphi'\mathfrak{F}\alpha$ is to be read as the composite of the natural transformation φ' with the natural transformation $\mathfrak{F}\alpha$.

Theorem: $\mathfrak{P}$ is right adjoint to $\mathfrak{Q}$ iff the functors $\mathrm{Hom}(\cdot,\mathfrak{P}\cdot)$ and $\mathrm{Hom}(\mathfrak{Q}\cdot,\cdot)$, which take (C,D) to $\mathrm{Hom}(C,\mathfrak{P}D)$ and $\mathrm{Hom}(\mathfrak{Q}C,D)$ respectively, and have the obvious action on maps, are naturally equivalent. [Proof: Suppose $\mathfrak{P}$ is right adjoint to $\mathfrak{Q}$. Then, for C an object of $\mathbf{C}$ and D an object of $\mathbf{D}$, we let $\theta(C,D)$ be the map which takes $\alpha \in \mathrm{Hom}(C,\mathfrak{P}D)$ to $\varphi = \varepsilon(D)\,\mathfrak{Q}\alpha$. Then $\theta(C,D)$ is bijective by the universal property of $\varepsilon(D)$ stated above. Moreover, θ is natural, because ε is. This shows that θ is a natural equivalence. Conversely, suppose $\theta\colon \mathrm{Hom}(\cdot,\mathfrak{P}\cdot) \cong \mathrm{Hom}(\mathfrak{Q}\cdot,\cdot)$ is a natural equivalence, and let $\varepsilon(D) = \theta(\mathfrak{P}D,D)(1_{\mathfrak{P}D})$ for every D. Then ε is natural. Moreover, if $\alpha\colon \mathfrak{G} \to \mathfrak{P}$ is a natural map (with $\mathfrak{G}$ a partial functor), then for each D we have, by naturality of θ, $\theta^{-1}(\mathfrak{G}D,D)(\varepsilon(D)\,\mathfrak{Q}\alpha(D)) = \theta^{-1}(\mathfrak{P}D,D)(\varepsilon(D))\,\alpha(D) = \alpha(D)$. So $\theta(\mathfrak{G}D,D)(\alpha(D)) = \varepsilon(D)\,\mathfrak{Q}\alpha(D)$. But then the bijectivity of θ immediately gives the required universal property.]

As a consequence, if $\mathfrak{P}$ is right adjoint to $\mathfrak{Q}$, then $\mathfrak{Q}$ is left adjoint to $\mathfrak{P}$. We write $\mathfrak{Q} \dashv \mathfrak{P}$ to say this.

To verify that $\mathfrak{P}$ is right adjoint to $\mathfrak{Q}$, it suffices to verify that the hypothetical counit $\varepsilon\colon \mathfrak{Q}\mathfrak{P} \to \mathfrak{I}$ is natural, and that it statisfies the required universal property for each object D of $\mathbf{D}$.

Let $\mathfrak{D}\colon \mathbf{Set} \rightsquigarrow \mathbf{Set} \times \mathbf{Set}$ be the diagonal functor, which sends every set A to (A,A) and every map f to (f,f). Then I claim that the right adjoint to $\mathfrak{F}$ exists and is the product functor $\mathfrak{P}\colon \mathbf{Set}\times\mathbf{Set} \rightsquigarrow \mathbf{Set}$. [Proof: Let $\mathfrak{P}$ be the product functor, and let $\varepsilon\colon \mathfrak{D}\mathfrak{P} \to \mathfrak{I}$ be the transformation which takes a pair (D,D') to the canonical map pair $(D \times D' \to D, D \times D' \to D')$. We must verify that ε is natural; but that is obvious. Now we must prove that if $(\varphi,\varphi')\colon \mathfrak{F}C \to (D,D')$ is a pair of maps of sets, then there exists a unique map $\alpha\colon C \to \mathfrak{P}(D,D')$ such that $(\varphi,\varphi') = \varepsilon(D,D')\mathfrak{F}\alpha$. But that is obvious: let $\alpha\colon C \to D \times D'$ be the unique map which projects to φ and φ'.]

One can also ask whether the diagonal functor $\mathfrak{D}$ defined above has a *left* adjoint. The answer is, yes, and that left adjoint $\mathfrak{Q}$ is the disjoint union functor. We leave it to the reader to formulate the exact statement. So we have a sequence of adjunctions

$$\mathfrak{Q} \dashv \mathfrak{D} \dashv \mathfrak{P}$$

The functor $\mathfrak{Q}$ does not have a left adjoint, and neither does the functor $\mathfrak{P}$ have a right adjoint.

On the other hand, consider the functor $A \times \cdot$ from $\mathbf{Set}$ to $\mathbf{Set}$, where A is a fixed set. This time, there is no left adjoint, but there is a right adjoint, namely the functor $\cdot^A$.

The forgetful functor $\mathfrak{G}\colon \mathbf{Group} \rightsquigarrow \mathbf{Set}$ has a left adjoint, the "free group functor" $\mathfrak{F}\colon \mathbf{Set} \rightsquigarrow \mathbf{Group}$. The same comment applies to all the "free X constructions" seen previously.

5. Presheafs and sheafs.

Let $\mathbf{C}$ be a category. If C is an object of $\mathbf{C}$, then a sieve on C is a class S of morphisms in $\mathbf{C}$ with range C such that if $f \in S$ and the range of g is the domain of f then $fg \in S$. The maximal sieve on C is the class of all morphisms in $\mathbf{C}$ with range C. If S is a sieve on C and $\varphi: D \to C$ is a morphism, then we define $\varphi^* S$ to be the collection of all morphisms f with range D such that $\varphi f \in S$. A Grothendieck topology on $\mathbf{C}$ is a map which assigns to each object C of $\mathbf{C}$ a collection $J(C)$ of sieves on C such that:

(i) The maximal sieve on C is in $J(C)$ for each C.

(ii) [Stability] If $S \in J(C)$ then $\varphi^*(S) \in J(D)$ for any morphism $\varphi: D \to C$.

(iii) [Transitivity] If $S \in J(C)$ and R is a sieve on C such that $\varphi^*(R) \in J(D)$ for all $\varphi: D \to C$ in S, then $R \in J(C)$.

If X is a topological space, and $\mathbf{O}$ the set of its open subsets, partially ordered by inclusion, and made into a category in the usual way, then a sieve on an object U of $\mathbf{O}$ is (can be identified with) a set of open subsets of U which is downward closed (it contains any open subset of any of its elements). We let $J(U)$ be the set of sieves on U whose elements cover U. It is obvious from the definition that this defines a Grothendieck topology on $\mathbf{O}$.

Let X be a separated scheme of finite type, and **Ét** the category whose objects are étale maps $U \to X$ with morphisms defined as in the case of the slice category. For $U \to X$ an étale morphism, let $J(U \to X)$ be the set of sieves S on $U \to X$ such that the images of the elements of S cover U. Then this defines a Grothendieck topology on **Ét**, called the étale topology on X.

Let $\mathbf{C}$ be a category with a Grothendieck topology J, and $\mathbf{D}$ be a category. Then a presheaf on $\mathbf{C}$ with values in $\mathbf{D}$ is just a *contravariant* functor from $\mathbf{C}$ to $\mathbf{D}$. If C is an object of $\mathbf{C}$, and $\mathcal{F}$ a presheaf on $\mathbf{C}$ with values in $\mathbf{D}$, and $S \in J(C)$, we define a matching family to be a family $(f_s)_{s \in S}$ of morphisms, where f_s has domain T (some object of $\mathbf{D}$, independent of s) and range $\mathcal{F}(D)$ where D is the domain of s, such that for any whenever $t: E \to D$ is a morphism in $\mathbf{C}$, and $s: D \to C$ is in S, we have $\mathcal{F}(t) f_s = f_{st}$; an amalgamation of such a family is a morphism $f: T \to \mathcal{F}(C)$ such that $\mathcal{F}(s) f = f_s$ for each $s \in J(C)$. If $\mathcal{F}$ is such that each matching family has a unique amalgamation, then we say that $\mathcal{F}$ is a sheaf on $\mathbf{C}$ with values in $\mathbf{D}$ for the Grothendieck topology J. Note that $\mathcal{F}(t) f_s$ above is generally written $f_s|_t$. Suppose $\mathbf{D}$ is the category of sets. Then to define a sheaf it is sufficient to consider matching families of *elements*, that, is, we can restrict ourselves to T being an (arbitrary) singleton, in which case f_s, for $s \in J(C)$, can be considered an element of $\mathcal{F}(D)$, where $s: D \to C$.

If, moreover, X is a topological space, and we define as above a Grothendieck topology on the set $\mathbf{O}$ of open subsets of X, then the condition reads: if S is a sieve covering the open set U, and for each $V \in S$ we have an element $x_V \in \mathcal{F}(V)$ such that $x_W = x_V|_W$ whenever $W \subseteq V$ is open, then there exists a unique $x \in \mathcal{F}(U)$ verifying $x_V = x|_U$ for each $V \subseteq U$ open.