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For $A \subseteq X$ and $B \subseteq Y$, consider the formula:

$$
\bigstar \quad \left(\begin{aligned} &\forall p:\Omega.\ \forall q:\Omega.\ ((\exists x:X.\ (x \in A \Rightarrow p)) \\ &\qquad \Rightarrow ((((\exists y:Y.\ (y \in B \Rightarrow q)) \Rightarrow q) \\ &\qquad\qquad \wedge (p \Rightarrow q)) \Rightarrow q)) \end{aligned} \right.
$$

Assertion: this is equivalent to the following:

$$
\heartsuit \quad \left(\begin{aligned} &\forall q:\Omega.\ ((((\exists y:Y.\ (y \in B \Rightarrow q)) \Rightarrow q) \\ &\qquad \wedge (\exists x:X.\ (x \in A \Rightarrow q))) \Rightarrow q) \end{aligned} \right.
$$

Alternatively:
$\forall q:\Omega.\ (((\exists y:Y.\ (y \in B \Rightarrow q)) \Rightarrow q)$
$\Rightarrow ((\exists x:X.\ (x \in A \Rightarrow q)) \Rightarrow q))$

Proof: $\boxed{\bigstar \Rightarrow \heartsuit}$ Assume $(\exists y:Y.\ (y \in B \Rightarrow q)) \Rightarrow q$
and $\exists x:X.\ (x \in A \Rightarrow q)$

By $\bigstar$ for $p = q$ we get q.
This shows $\heartsuit$. $\square$

$\boxed{\heartsuit \Rightarrow \bigstar}$ Assume $\exists x:X.\ (x \in A \Rightarrow p)$, say $x_0 \in A \Rightarrow p$,
and assume $(\exists y:Y.\ (y \in B \Rightarrow q)) \Rightarrow q$
and $p \Rightarrow q$

In particular, $x_0 \in A \Rightarrow q$, so $\exists x:X.\ (x \in A \Rightarrow q)$

By $\heartsuit$ we get q. This shows $\bigstar$. $\square$

N.B.: Calling $\phi_A(p) :=$ «$\exists x:X.\ (x \in A \Rightarrow p)$», and noting $(p_1 \Rightarrow p_2) \Rightarrow (\phi_A(p_1) \Rightarrow \phi_A(p_2))$
the smallest nucleus $\geq \phi_A$ is given by $j_A(p) :=$ «$\forall q:\Omega.\ (((\phi_A(q) \Rightarrow q) \wedge (p \Rightarrow q)) \Rightarrow q)$»

So $\bigstar$ says $\phi_A \leq j_B$, or equivalently $j_A \leq j_B$.
(Note j_A is also the smallest nucleus j s.t. $\forall x:X.\ (j(x \in A))$.)

This will be of particular use when applied to $D(A)$, $D(B)$ (instead of A, B)
given by $x \in D(A) :\Leftrightarrow (x \in A \vee \neg x \in A)$.
So $j_{D(A)}$ is the smallest nucleus deciding A.